Vector base continuum

non-cauchian theorem

part 1

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Introduction:

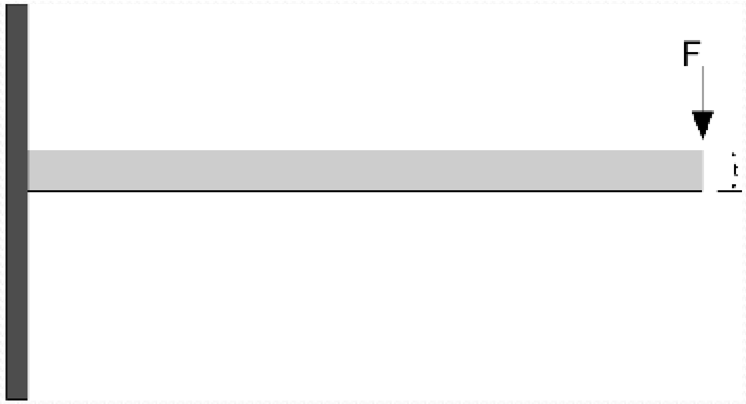
Symmetry of stress tensor ($\tau_{xy} = \tau_{yx}$) is the start point of this survey. As we know in non-magnetic fields this symmetry is always valid and everybody has been accepting that for 200 years after Cauchy.

While in quantum or discrete mechanics (statistical mechanics) it's not evident. Also there are some continua situations, in them, that symmetry isn't valid. So what is the reason of this difference? What is the problem?

Attention:

The nonsymmetrical result obtained here isn’t due to body or surface moments such as a magnetic field and it’s valid for all continua.

Counterexample:



$\tau_{xy} = 0$

And $\bar{\tau}_{yx}.t = F \quad \Rightarrow \quad \bar{\tau}_{yx} = \frac{F}{t}$

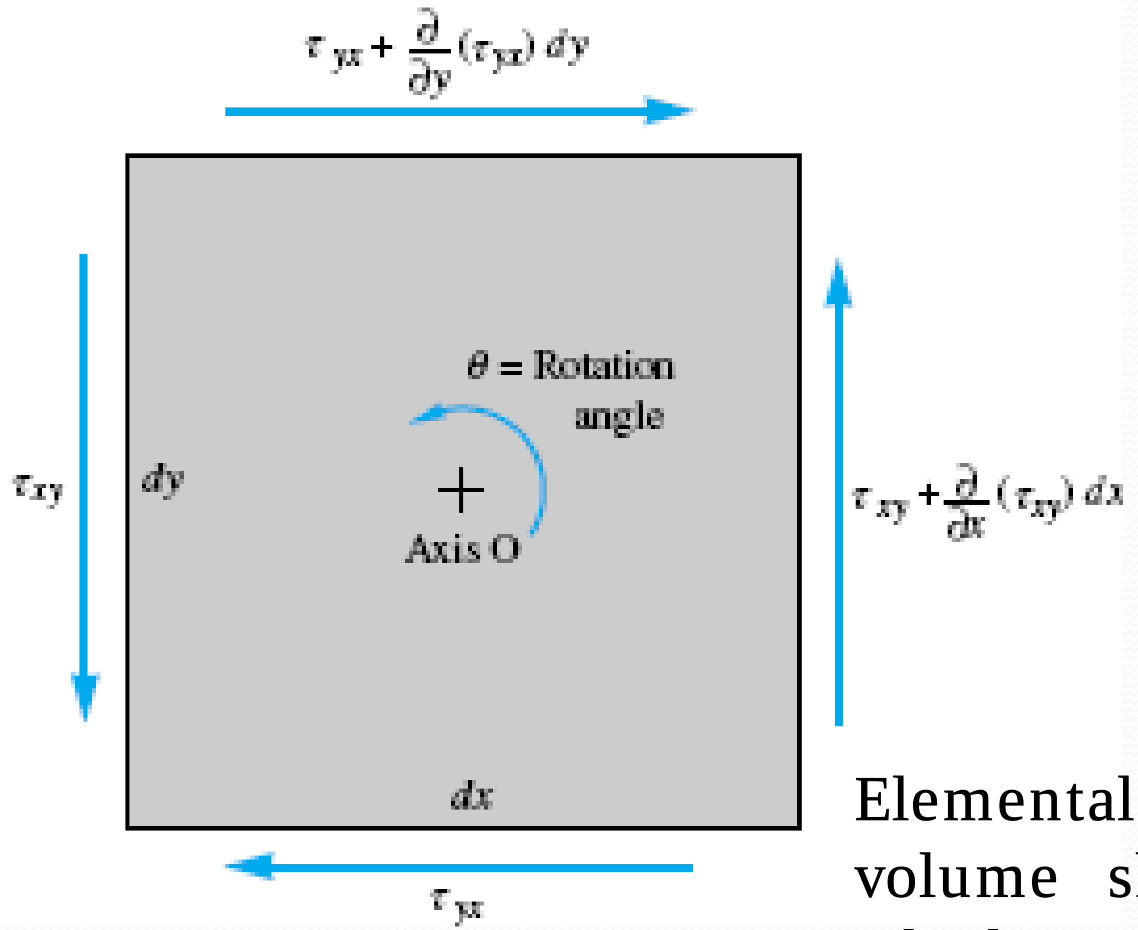
If $t, F \to 0$ *then* $\quad \tau_{yx} = \lim_{t,F\to 0} \frac{F}{t} \neq 0$

So in this case $\tau_{xy} \neq \tau_{yx}$

Prove of symmetry:

There are two different ways, that the first (in fluid dynamic books) is worse than the other one:

1. Using a differential cubic element as control volume and applying the Angular Momentum conservation on it. Then terms of angular inertia become so small (higher order differential) and neglectable. The result is the angular equilibrium of the cube around each axis that force the shear stresses to be same.



Elemental Cartesian fixed control volume showing shear stresses which may cause a net angular acceleration about axis o.

Prove of symmetry...

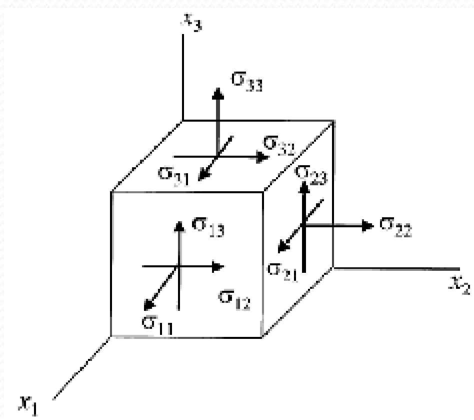
$$\sum M_o = \frac{\partial}{\partial t}\left[\int_{CV} (r \times V)\rho . dv\right] + \int_{CS} (r \times V)\rho(V.n)dA$$

$$\left[\tau_{xy} - \tau_{yx} + \frac{1}{2}\frac{\partial}{\partial x}(\tau_{xy})dx - \frac{1}{2}\frac{\partial}{\partial y}(\tau_{yx})dy\right] dxdydz = \frac{1}{12}\rho(dxdydz)(dx^2 + dy^2)\frac{d^2\theta}{dt^2}$$

$$dxdydz \gg dx^2 dydz, dxdy^2 dz \gg dx^3 dydz, dxdy^3 dz$$

$$\tau_{xy} \approx \tau_{yx}$$

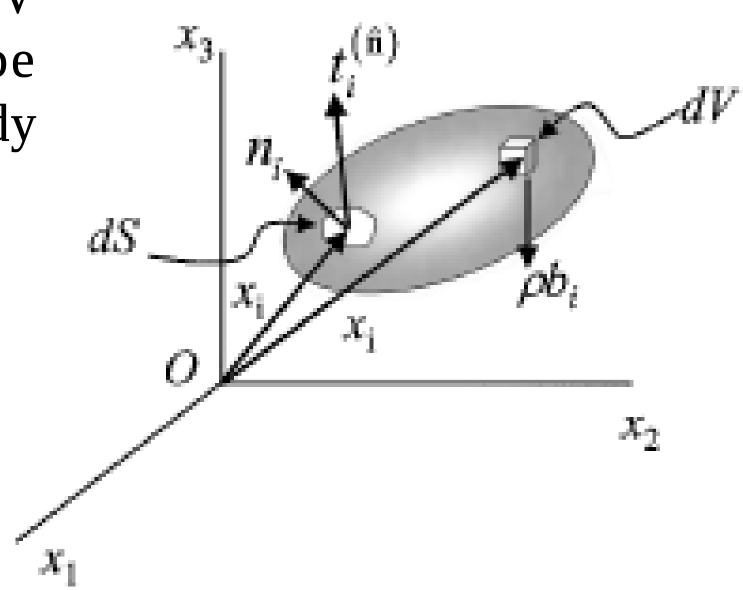
Please notice to The approximate result and the differential cubic element!



Prove of symmetry...

2. Using an arbitrary control volume and applying all conservation laws (mass, Linear & Angular Momentum) on it.

Consider a material body having a volume V and a boundary surface S. let the body be subjected to surface tractions $t_i^{(\hat{n})}$ and body forces b_i (force per unit mass)



Prove of symmetry...

force Equilibrium:

$$\int_S t_i^{(\hat{\mathbf{n}})} dS + \int_V \rho b_i dV = 0$$

where dS is the differential element of the surface S and dV that of volume V.

If we admit $t_i^{(\hat{\mathbf{n}})} = \sigma_{ji} n_j$ then

$$\int_S \sigma_{ji} n_j dS = \int_V \sigma_{ji,j} dV$$

From divergence theorem: $\int_V \left(\sigma_{ji,j} + \rho b_i\right) dV = 0$ $\quad \sigma_{ji,j} + \rho b_i = 0$

Prove of symmetry...

Balance of moment:

$$\int_S \varepsilon_{ijk} x_j t_k^{(\hat{\mathbf{n}})} dS + \int_V \varepsilon_{ijk} x_j \rho b_k dV = 0$$

using the identity $t_k^{(\hat{\mathbf{n}})} = \sigma_{qk} n_q$ and Gauss's divergence theorem, we obtain:

$$\int_V \varepsilon_{ijk} \left[\left(x_j \sigma_{qk} \right)_{,q} + x_j \rho b_k \right] dV = 0$$

$$\int_V \varepsilon_{ijk} \left[x_{j,q} \sigma_{qk} + x_j \left(\sigma_{qk,q} + \rho b_k \right) \right] dV = 0$$

Prove of symmetry...

But $x_{j,q} = \delta_{jq}$ and $\sigma_{qk,q} + \rho b_k = 0$ so

$$\int_V \varepsilon_{ijk}\sigma_{jk}\, dV = 0$$

$$\varepsilon_{ijk}\sigma_{jk} = 0$$

For i=1 $\varepsilon_{123}\sigma_{23} + \varepsilon_{132}\sigma_{32} = 0$, or $\sigma_{23} - \sigma_{32} = 0$

$$\sigma_{jk} = \sigma_{kj}$$

Prove of symmetry…

Nothing wrong, both ways give same outputs, so we have to return assumptions because the point isn't in this stage. assumptions such as Newton laws and the relation between stress and orientation (Cauchy principles).

We are not at velocity about light velocity or in a quantum field so Newton law is valid . So we are going to check out Cauchy principles.

Stress and direction relation:

Stress is a function of position and direction so:

$$\vec{\sigma} = t^{(\hat{n})} = \Sigma(\vec{r}, \hat{n}) : \mathbb{R}^5 \to \mathbb{R}^3$$

This sigma must obey some rules.

1. the third Newton law is led to:

$$\Sigma(\vec{r}, -\hat{n}) = -\Sigma(\vec{r}, \hat{n})$$

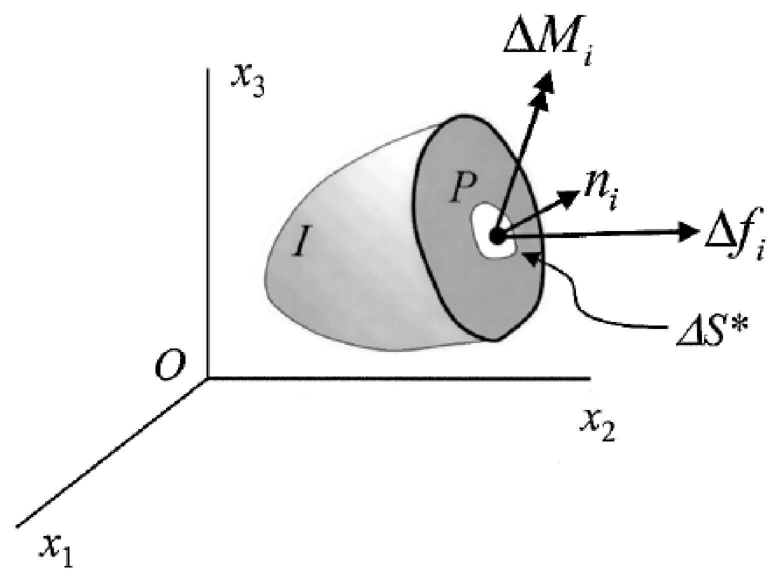
2. sigma is a conservative and analytic function of direction, the same as Cauchy–Riemann differential equations in complex analysis. Later we show this one may be led to vector form. Sigma is invariant from orientation and displacement of global frame. This rule is not as clear as others and maybe it's not!

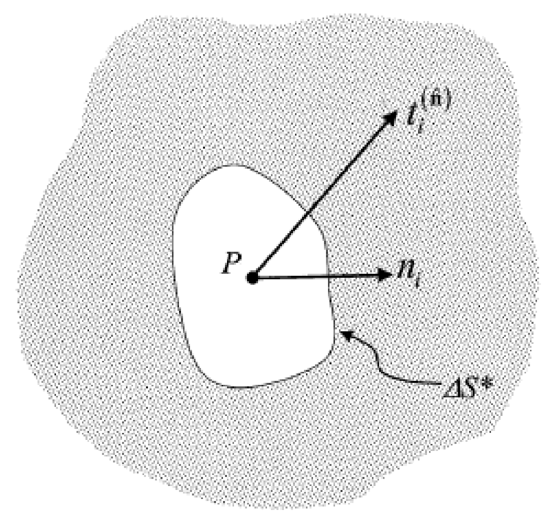
3. The same results should be obtained from linear momentum conservation and angular conservation. That means the following relation can be valid for any arbitrary control volume:

$$\int_{cs} \left(\vec{r} \times \Sigma(\vec{r}, \hat{n})\right) dA = \vec{r} \times \int_{cs} \Sigma(\vec{r}, \hat{n}) dA$$

Cauchy principles:

Cauchy used a tetrahedron element and apply linear equilibrium to obtain the relation between sigma and n.





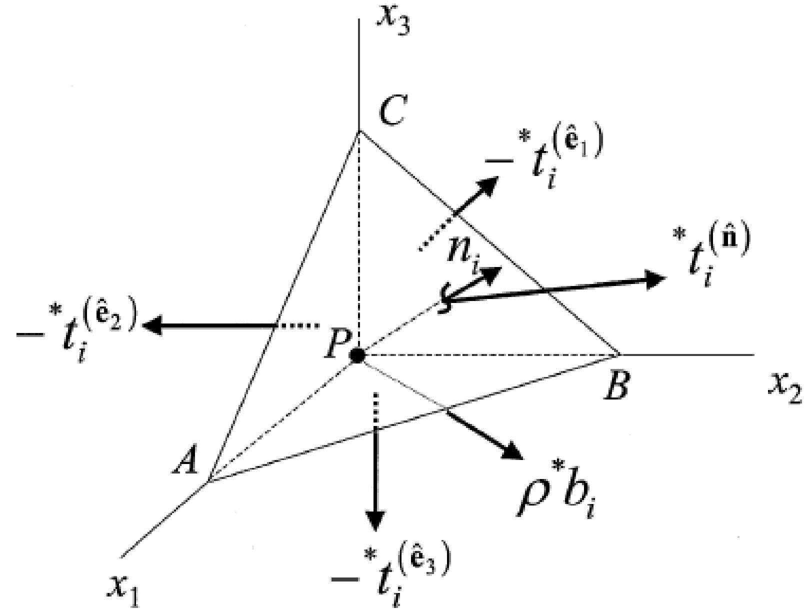
$$ {}^*t_i^{(\hat{\mathbf{n}})} dS - {}^*t_i^{(\hat{\mathbf{e}}_j)} n_j dS + \rho^* b_i dV = 0 $$

$$ dV = \frac{1}{3} h dS $$

$$ {}^*t_i^{(\hat{\mathbf{n}})} = {}^*t_i^{(\hat{\mathbf{e}}_j)} n_j - \frac{1}{3} \rho^* b_i h $$

$$ h \to 0 $$

$$ t_i^{(\hat{\mathbf{n}})} = t_i^{(\hat{\mathbf{e}}_j)} n_j $$



$$ \sigma_{ji} \equiv t_i^{(\hat{\mathbf{e}}_j)} $$

$$ t_i^{(\hat{\mathbf{n}})} = \sigma_{ji} n_j \quad \text{or} \quad \mathbf{t}^{(\hat{\mathbf{n}})} = \hat{\mathbf{n}} \cdot \boldsymbol{\sigma} $$

Cauchy principles …

Why a differential tetrahedron element is used? Nobody cares. But it's the point because this element just differentiate the position not the orientation!

And this result is not valid for any arbitrary volume, we will show the Cauchy principle is just a approximate solution of sigma. However we don't know the exact one but it doesn't matter.

Cauchy stress formula is a approximation:

Consider a linear approximation relation between sigma and n. it can be illustrated as a 3*3 matrix:

$$\vec{\sigma} = \check{T}.\hat{n}$$

If i, j, k are substituted instead of n, a interesting result may be obtained :

$$\check{T}.\hat{\imath} = \begin{bmatrix} T_{xx} \\ T_{xy} \\ T_{xz} \end{bmatrix} = t^{(\hat{\imath})}$$

$$\check{T} = \check{\sigma}$$

It can be found without the complicated mathematics, the same result was obtained. So it may be claimed that a new rule in mechanic is found!

But it's nonsensical to say fortunately the exact solution is linear as approximate one!

we have to admit the principle of Cauchy is just a approximation for sigma.

Next:

In next we will try to find the most general shape for the relation. It's just a claim to say vector form of equations is a valid shape for Cauchy–Riemann rules and we can't prove it yet.

We will generate the conservative rule equations without assumption of linear mapping. And we show the computational cost advantages of vector base continuum in compare whit tensor.

Some constitutive equations will be written in vector base form and we regenerate some well-known deformation such as navier-stockse