a) su(2) Algebra Representations

Define a basis for $su(2)$ as:

$$u_1 = i\sigma_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad u_2 = i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad u_3 = i\sigma_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}. \tag{1}$$

We have

$$[u_1, u_2] = -2u_3, \quad [u_2, u_3] = -2u_1, \quad [u_3, u_1] = -2u_2 \tag{2}$$

First we compute $[(\alpha_1, \alpha_2, \alpha_3), (\alpha'_1, \alpha'_2, \alpha'_3)]$ defined by $[(\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3), (\alpha'_1 u_1 + \alpha'_2 u_2 + \alpha'_3 u_3)]$.

$$\begin{aligned} & (\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3)(\alpha'_1 u_1 + \alpha'_2 u_2 + \alpha'_3 u_3) \\ = \; & -(\alpha_1\alpha'_1 + \alpha_2\alpha'_2 + \alpha_3\alpha'_3)\mathbb{1} + \alpha_1\alpha'_2 u_1 u_2 + \alpha_2\alpha'_1 u_2 u_1 \\ & +\alpha_1\alpha'_3 u_1 u_3 + \alpha_3\alpha'_1 u_3 u_1 + \alpha_2\alpha'_3 u_2 u_3 + \alpha_3\alpha'_2 u_3 u_2. \end{aligned} \tag{3}$$

So that

$$\begin{aligned} & [(\alpha_1, \alpha_2, \alpha_3), (\alpha'_1, \alpha'_2, \alpha'_3)] \\ = \; & \alpha_1\alpha'_2(u_1u_2 - u_2u_1) + \alpha_2\alpha'_1(u_2u_1 - u_1u_2) \\ & \alpha_1\alpha'_3(u_1u_3 - u_3u_1) + \alpha_3\alpha'_1(u_3u_1 - u_1u_3) \\ & +\alpha_2\alpha'_3(u_2u_3 - u_3u_2) + \alpha_3\alpha'_2(u_3u_2 - u_2u_3) \\ = \; & 2(\alpha_3\alpha'_2 - \alpha_2\alpha'_3)u_1 + 2(\alpha_1\alpha'_3 - \alpha_3\alpha'_1)u_2 + 2(\alpha_2\alpha'_1 - \alpha_1\alpha'_2)u_3 \end{aligned} \tag{4}$$

which we will use in the following.

From

$$\begin{aligned} \varphi(\alpha_1\sigma_1 + \alpha_2\sigma_2 + \alpha_3\sigma_3) \quad & . \quad (a_0 + a_1x + a_2x^2 + a_3y + a_4y^2 + a_5xy) \\ & = \; x(-i\alpha_1a_3 + \alpha_2a_3 - \alpha_3a_1) + \\ & + \; x^2(2i\alpha_1a_5 + 2\alpha_2a_5 + 2\alpha_3a_2) + \\ & + \; y(i\alpha_1a_1 + \alpha_2a_1 + \alpha_3a_3) + \\ & + \; y^2(2i\alpha_1a_5 - 2\alpha_2a_5 - 2\alpha_3a_4) + \\ & + \; xy(-i\alpha_1a_2 - i\alpha_1a_4 + \alpha_2a_2 - \alpha_2a_4) \end{aligned} \tag{5}$$

we have for the adjusted φ:

$$\begin{array}{rcl}
\tilde{\varphi}(\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3) & . & (a_0 + a_1 x + a_2 x^2 + a_3 y + a_4 y^2 + a_5 xy) \\
\varphi(\alpha_1(i\sigma_1) + \alpha_2(i\sigma_2) + \alpha_3(i\sigma_3)) & . & (a_0 + a_1 x + a_2 x^2 + a_3 y + a_4 y^2 + a_5 xy) \\
\varphi((i\alpha_1)\sigma_1 + (i\alpha_2)\sigma_2 + (i\alpha_3)\sigma_3) & . & (a_0 + a_1 x + a_2 x^2 + a_3 y + a_4 y^2 + a_5 xy) \\
& = & x(\alpha_1 a_3 + i\alpha_2 a_3 - i\alpha_3 a_1) + \\
& + & x^2(-2\alpha_1 a_5 + 2i\alpha_2 a_5 + 2i\alpha_3 a_2) + \\
& + & y(-\alpha_1 a_1 + i\alpha_2 a_1 + i\alpha_3 a_3) + \\
& + & y^2(-2\alpha_1 a_5 - 2i\alpha_2 a_5 - 2i\alpha_3 a_4) + \\
& + & xy(\alpha_1 a_2 + \alpha_1 a_4 + i\alpha_2 a_2 - i\alpha_2 a_4)
\end{array} \tag{6}$$

and from this we can read off:

$$\begin{array}{rcl}
\tilde{\varphi}(\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3) & . & 1 = 0 \\
\tilde{\varphi}(\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3) & . & x = -xi\alpha_3 + y(-\alpha_1 + i\alpha_2) \\
\tilde{\varphi}(\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3) & . & x^2 = 2x^2 i\alpha_3 - xy(-\alpha_1 - i\alpha_2) \\
\tilde{\varphi}(\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3) & . & y = x(-\alpha_1 + i\alpha_2) + yi\alpha_3 \\
\tilde{\varphi}(\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3) & . & y^2 = -2iy^2\alpha_3 - xy(-\alpha_1 + i\alpha_2) \\
\tilde{\varphi}(\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3) & . & xy = 2x^2(-\alpha_1 + i\alpha_2) + 2y^2(-\alpha_1 - i\alpha_2).
\end{array} \tag{7}$$

We will use these in the following to prove

$$[\tilde{\varphi}(\alpha_1, \alpha_2, \alpha_3), \tilde{\varphi}(\alpha_1', \alpha_2', \alpha_3',)] = \tilde{\varphi}([(\alpha_1, \alpha_2, \alpha_3), (\alpha_1', \alpha_2', \alpha_3')])$$

when applied to each of $1, x, x^2, y, y^2$ and xy individually.

The case of 1

Take the case of 1. Obviously:

$$[\tilde{\varphi}(\alpha_1, \alpha_2, \alpha_3), \tilde{\varphi}(\alpha_1', \alpha_2', \alpha_3',)] \;.\; 1 = 0 \tag{8}$$

and then from (4) and (7) we obviously have:

$$\tilde{\varphi}[(\alpha_1, \alpha_2, \alpha_3), (\alpha_1', \alpha_2', \alpha_3')] \;.\; 1 = 0. \tag{9}$$

The case of x

Take the case of x. We will need:

$$\begin{array}{rcl}
\tilde{\varphi}(\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3) & . & x = -xi\alpha_3 + y(-\alpha_1 + i\alpha_2) \\
\tilde{\varphi}(\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3) & . & y = x(\alpha_1 + i\alpha_2) + yi\alpha_3 \\
[(\alpha_1, \alpha_2, \alpha_3), (\alpha_1', \alpha_2', \alpha_3')] & = & 2(\alpha_3\alpha_2' - \alpha_2\alpha_3')u_1 + 2(\alpha_1\alpha_3' - \alpha_3\alpha_1')u_2 + 2(\alpha_2\alpha_1' - \alpha_1\alpha_2')u_3
\end{array} \tag{10}$$

First

$$\begin{array}{rl}
& \tilde{\varphi}(\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3)\tilde{\varphi}(\alpha_1' u_1 + \alpha_2' u_2 + \alpha_3' u_3) \;.\; x \\
= & \tilde{\varphi}(\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3) \;.\; \big(-xi\alpha_3' + y(-\alpha_1' + i\alpha_2')\big) \\
= & -(-xi\alpha_3 + y(-\alpha_1 + i\alpha_2))i\alpha_3' + (x(\alpha_1 + i\alpha_2) + yi\alpha_3)(-\alpha_1' + i\alpha_2') \\
= & -iy(-\alpha_1 + i\alpha_2)\alpha_3' + x(\alpha_1 + i\alpha_2)(-\alpha_1' + i\alpha_2') + yi\alpha_3(-\alpha_1' + i\alpha_2') \\
= & -xi(\alpha_2\alpha_1' - \alpha_1\alpha_2') - y(\alpha_3\alpha_2' - \alpha_2\alpha_3') + yi(\alpha_1\alpha_3' - \alpha_3\alpha_1') + \ldots
\end{array} \tag{11}$$

so that

$$\begin{array}{rl}
& [\tilde{\varphi}(\alpha_1, \alpha_2, \alpha_3), \tilde{\varphi}(\alpha_1', \alpha_2', \alpha_3')] \;.\; x \\
= & -x2i(\alpha_2\alpha_1' - \alpha_1\alpha_2') + y(-2(\alpha_3\alpha_2' - \alpha_2\alpha_3') + 2i(\alpha_1\alpha_3' - \alpha_3\alpha_1'))
\end{array} \tag{12}$$

and then from (4) and (7) we have

$$\begin{array}{rl}
& \tilde{\varphi}([(\alpha_1, \alpha_2, \alpha_3), (\alpha_1', \alpha_2', \alpha_3')]) \;.\; x \\
= & -xi2(\alpha_2\alpha_1' - \alpha_1\alpha_2') + y(-2(\alpha_3\alpha_2' - \alpha_2\alpha_3') + i2(\alpha_1\alpha_3' - \alpha_3\alpha_1'))
\end{array} \tag{13}$$

The case of x^2

Take the case of x^2. We will need:

$$\begin{aligned}
\tilde{\varphi}(\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3) &\ . \ \ x^2 = 2x^2 i\alpha_3 + xy(\alpha_1 + i\alpha_2) \\
\tilde{\varphi}(\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3) &\ . \ \ xy = 2x^2(-\alpha_1 + i\alpha_2) - 2y^2(\alpha_1 + i\alpha_2) \\
[(\alpha_1, \alpha_2, \alpha_3), (\alpha'_1, \alpha'_2, \alpha'_3)] &= 2(\alpha_3\alpha'_2 - \alpha_2\alpha'_3)u_1 + 2(\alpha_1\alpha'_3 - \alpha_3\alpha'_1)u_2 + 2(\alpha_2\alpha'_1 - \alpha_1\alpha'_2)u_3
\end{aligned} \tag{14}$$

First

$$\begin{aligned}
& \tilde{\varphi}(\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3)\tilde{\varphi}(\alpha'_1 u_1 + \alpha'_2 u_2 + \alpha'_3 u_3) \ . \ x^2 \\
= \ & \tilde{\varphi}(\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3) \ . \ \big(2x^2 i\alpha'_3 + xy(\alpha'_1 + i\alpha'_2)\big) \\
= \ & 2\big(x^2 i\alpha_3 + xy(\alpha_1 + i\alpha_2)\big)i\alpha'_3 + \big(2x^2(-\alpha_1 + i\alpha_2) - 2y^2(\alpha_1 + i\alpha_2)\big)(\alpha'_1 + i\alpha'_2) \\
= \ & 2xy(\alpha_1 + i\alpha_2)i\alpha'_3 + 2x^2(-\alpha_1 + i\alpha_2)(\alpha'_1 + i\alpha'_2) \\
= \ & 2x^2 i(\alpha_2\alpha'_1 - \alpha_1\alpha'_2) + xy\big(-2\alpha_2\alpha'_3 + 2i\alpha_1\alpha'_3\big)
\end{aligned} \tag{15}$$

so that

$$\begin{aligned}
& [\tilde{\varphi}(\alpha_1, \alpha_2, \alpha_3), \tilde{\varphi}(\alpha'_1, \alpha'_2, \alpha'_3)] \ . \ x^2 \\
= \ & 2x^2 i2(\alpha_2\alpha'_1 - \alpha_1\alpha'_2) + xy\big(2(\alpha_3\alpha'_2 - \alpha_2\alpha'_3) + 2i(\alpha_1\alpha'_3 - \alpha_3\alpha'_1)\big)
\end{aligned} \tag{16}$$

and then from (4) and (7) we have

$$\begin{aligned}
& \tilde{\varphi}([(\alpha_1, \alpha_2, \alpha_3), (\alpha'_1, \alpha'_2, \alpha'_3)]) \ . \ x^2 \\
= \ & 2x^2 i2(\alpha_2\alpha'_1 - \alpha_1\alpha'_2) + xy(2(\alpha_3\alpha'_2 - \alpha_2\alpha'_3) + i2(\alpha_1\alpha'_3 - \alpha_3\alpha'_1))
\end{aligned} \tag{17}$$

The case of y

Take the case of y. We will need:

$$\begin{aligned}
\tilde{\varphi}(\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3) \quad &. \quad x = -xi\alpha_3 + y(-\alpha_1 + i\alpha_2) \\
\tilde{\varphi}(\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3) \quad &. \quad y = x(\alpha_1 + i\alpha_2) + yi\alpha_3 \\
[(\alpha_1, \alpha_2, \alpha_3), (\alpha'_1, \alpha'_2, \alpha'_3)] \quad &= \quad 2(\alpha_3\alpha'_2 - \alpha_2\alpha'_3)u_1 + 2(\alpha_1\alpha'_3 - \alpha_3\alpha'_1)u_2 + 2(\alpha_2\alpha'_1 - \alpha_1\alpha'_2)u_3
\end{aligned} \tag{18}$$

First

$$\begin{aligned}
& \tilde{\varphi}(\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 i_3)\varphi(\alpha'_1 u_1 + \alpha'_2 u_2 + \alpha'_3 u_3) \;.\; y \\
= \;& \tilde{\varphi}(\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3) \;.\; \big(x(\alpha'_1 + i\alpha'_2) + yi\alpha'_3\big) \\
= \;& \big(-xi\alpha_3 + y(-\alpha_1 + i\alpha_2)\big)(\alpha'_1 + i\alpha'_2) + i\big(x(\alpha_1 + i\alpha_2) + yi\alpha_3\big)\alpha'_3 \\
= \;& -xi\alpha_3(\alpha'_1 + i\alpha'_2) + xi(\alpha_1 + i\alpha_2)\alpha'_3 + y(-\alpha_1 + i\alpha_2)(\alpha'_1 + i\alpha'_2) + \ldots \\
= \;& x(\alpha_3\alpha'_2 - \alpha_2\alpha'_3) + xi(\alpha_1\alpha'_3 - \alpha_3\alpha'_1) + yi(\alpha_2\alpha'_1 - \alpha_1\alpha'_2) + \ldots
\end{aligned} \tag{19}$$

so that

$$\begin{aligned}
& [\tilde{\varphi}(\alpha_1, \alpha_2, \alpha_3), \tilde{\varphi}(\alpha'_1, \alpha'_2, \alpha'_3)] \;.\; y \\
= \;& x2(\alpha_3\alpha'_2 - \alpha_2\alpha'_3) + xi2(\alpha_1\alpha'_3 - \alpha_3\alpha'_1) + yi2(\alpha_2\alpha'_1 - \alpha_1\alpha'_2)
\end{aligned} \tag{20}$$

and then from (4) and (7) we have

$$\begin{aligned}
& \tilde{\varphi}([(\alpha_1, \alpha_2, \alpha_3), (\alpha'_1, \alpha'_2, \alpha'_3)]) \;.\; y \\
= \;& x\big(2(\alpha_3\alpha'_2 - \alpha_2\alpha'_3) + i2(\alpha_1\alpha'_3 - \alpha_3\alpha'_1)\big) + yi2(\alpha_2\alpha'_1 - \alpha_1\alpha'_2)
\end{aligned} \tag{21}$$

The case of y^2

Take the case of y^2. We will need:

$$\begin{aligned}
\tilde{\varphi}(\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3) &\ . \ y^2 = -2iy^2\alpha_3 - xy(-\alpha_1 + i\alpha_2) \\
\tilde{\varphi}(\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3) &\ . \ xy = 2x^2(-\alpha_1 + i\alpha_2) - 2y^2(\alpha_1 + i\alpha_2) \\
[(\alpha_1, \alpha_2, \alpha_3), (\alpha_1', \alpha_2', \alpha_3')] &= 2(\alpha_3\alpha_2' - \alpha_2\alpha_3')u_1 + 2(\alpha_1\alpha_3' - \alpha_3\alpha_1')u_2 + 2(\alpha_2\alpha_1' - \alpha_1\alpha_2')u_3
\end{aligned} \tag{22}$$

First

$$\begin{aligned}
& \tilde{\varphi}(\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 i_3)\varphi(\alpha_1' u_1 + \alpha_2' u_2 + \alpha_3' u_3) \ . \ y^2 \\
= \ & \tilde{\varphi}(\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3) \ . \ \big(-2iy^2\alpha_3' - xy(-\alpha_1' + i\alpha_2')\big) \\
= \ & -2i\big(-2iy^2\alpha_3 - xy(-\alpha_1 + i\alpha_2)\big)\alpha_3' - \big(2x^2(-\alpha_1 + i\alpha_2) + 2y^2(-\alpha_1 - i\alpha_2)\big)(-\alpha_1' + i\alpha_2') \\
= \ & -2y^2(-\alpha_1 - i\alpha_2)(-\alpha_1' + i\alpha_2') + xy2i(-\alpha_1 + i\alpha_2)\alpha_3' \\
= \ & -2y^2 i(\alpha_2\alpha_1' - \alpha_1\alpha_2') - xy\big(2\alpha_2\alpha_3' + i2\alpha_1\alpha_3'\big)
\end{aligned} \tag{23}$$

so that

$$\begin{aligned}
& [\tilde{\varphi}(\alpha_1, \alpha_2, \alpha_3), \tilde{\varphi}(\alpha_1', \alpha_2', \alpha_3')] \ . \ y^2 \\
= \ & -2y^2 i2(\alpha_2\alpha_1' - \alpha_1\alpha_2') - xy(-2(\alpha_3\alpha_2' - \alpha_2\alpha_3') + i2(\alpha_1\alpha_3' - \alpha_3\alpha_1'))
\end{aligned} \tag{24}$$

and then from (4) and (7) we have

$$\begin{aligned}
& \tilde{\varphi}([(\alpha_1, \alpha_2, \alpha_3), (\alpha_1', \alpha_2', \alpha_3')]) \ . \ y^2 \\
= \ & -2iy^2 2(\alpha_2\alpha_1' - \alpha_1\alpha_2') - xy(-2(\alpha_3\alpha_2' - \alpha_2\alpha_3') + i2(\alpha_1\alpha_3' - \alpha_3\alpha_1'))
\end{aligned} \tag{25}$$

The case of xy

Take the case of xy. We will need:

$$\begin{aligned}
\tilde{\varphi}(\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3) &\ . \ xy = 2x^2(-\alpha_1 + i\alpha_2) - 2y^2(\alpha_1 + i\alpha_2) \\
\tilde{\varphi}(\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3) &\ . \ x^2 = 2x^2 i\alpha_3 + xy(\alpha_1 + i\alpha_2) \\
\tilde{\varphi}(\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3) &\ . \ y^2 = -2iy^2\alpha_3 - xy(-\alpha_1 + i\alpha_2) \\
[(\alpha_1, \alpha_2, \alpha_3), (\alpha_1', \alpha_2', \alpha_3')] &= 2(\alpha_3\alpha_2' - \alpha_2\alpha_3')u_1 + 2(\alpha_1\alpha_3' - \alpha_3\alpha_1')u_2 + 2(\alpha_2\alpha_1' - \alpha_1\alpha_2')u_3
\end{aligned} \tag{26}$$

First

$$\begin{aligned}
& \tilde{\varphi}(\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 i_3)\varphi(\alpha_1' u_1 + \alpha_2' u_2 + \alpha_3' u_3) \ . \ xy \\
= \ & \tilde{\varphi}(\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3) \ . \ \big(2x^2(-\alpha_1' + i\alpha_2') + 2y^2(-\alpha_1' - i\alpha_2')\big) \\
= \ & 2\big(2x^2 i\alpha_3 + xy(\alpha_1 + i\alpha_2)\big)(-\alpha_1' + i\alpha_2') + 2\Big(- 2iy^2\alpha_3 - xy(-\alpha_1 + i\alpha_2)\Big)(-\alpha_1' - i\alpha_2') \\
= \ & 2x^2 2i\alpha_3(-\alpha_1' + i\alpha_2') + 2y^2 2i\alpha_3(\alpha_1' + i\alpha_2') \\
= \ & 2x^2(-2\alpha_3\alpha_2' - 2i\alpha_3\alpha_1') + 2y^2(-2\alpha_3\alpha_2' - 2(-i\alpha_3\alpha_1'))
\end{aligned} \tag{27}$$

so that

$$\begin{aligned}
& [\tilde{\varphi}(\alpha_1, \alpha_2, \alpha_3), \tilde{\varphi}(\alpha_1', \alpha_2', \alpha_3')] \ . \ xy \\
= \ & 2x^2\Big(- 2(\alpha_3\alpha_2' - \alpha_2\alpha_3') + i2(\alpha_1\alpha_3' - \alpha_3\alpha_1')\Big) + 2y^2\Big(- 2(\alpha_3\alpha_2' - \alpha_2\alpha_3') - 2i(\alpha_1\alpha_3' - \alpha_3\alpha_1')\Big)
\end{aligned} \tag{28}$$

and then from (4) and (7) we have

$$\begin{aligned}
& \tilde{\varphi}([(\alpha_1, \alpha_2, \alpha_3), (\alpha_1', \alpha_2', \alpha_3')]) \ . \ xy \\
= \ & 2x^2\Big(- 2(\alpha_3\alpha_2' - \alpha_2\alpha_3') + i2(\alpha_1\alpha_3' - \alpha_3\alpha_1')\Big) + 2y^2\Big(- 2(\alpha_3\alpha_2' - \alpha_2\alpha_3') - 2i(\alpha_1\alpha_3' - \alpha_3\alpha_1')\Big)
\end{aligned} \tag{29}$$

b) Irreducible components:

Components that are transformed into linear combinations of themselves under repeated application of the operators $\tilde{\varphi}(\alpha_1 u_1 + u_2\sigma_2 + \alpha_3 u_3)$ form an invariant subspace. Irreducible components are an invariant subspace which cannot be separated into smaller invariant subspaces.

It is easy to see from (7) that the irreducible components are:

$$\begin{aligned} &\{1\} \\ &\{x, y\} \\ &\{y^2, xy, x^2\} \end{aligned} \tag{30}$$

c) Vectors of maximal weight:

To determine the weights, we consider

$$H = \sigma_3, \quad X = \frac{1}{2}\sigma_1 + \frac{1}{2}i\sigma_2, \quad Y = \frac{1}{2}\sigma_1 - \frac{1}{2}i\sigma_2 \tag{31}$$

or

$$H = -iu_3, \quad X = -\frac{i}{2}u_1 + \frac{1}{2}u_2, \quad Y = -\frac{i}{2}u_1 - \frac{1}{2}u_2 \tag{32}$$

Bracket algebra of H, X, and Y

We have the algebraic relations

$$\begin{aligned} [\frac{1}{2}H, X] &= [\frac{1}{2}\sigma_3, \frac{1}{2}\sigma_1 + \frac{1}{2}i\sigma_2] = \frac{i}{2}\sigma_2 + \frac{1}{2}\sigma_1 = X, \\ [\frac{1}{2}H, Y] &= [\frac{1}{2}\sigma_3, \frac{1}{2}\sigma_1 - \frac{1}{2}i\sigma_2] = -Y \\ [X, Y] &= [\frac{1}{2}\sigma_1 + \frac{1}{2}i\sigma_2, \frac{1}{2}\sigma_1 - \frac{1}{2}i\sigma_2] = \sigma_3 = 2\frac{1}{2}H. \end{aligned} \tag{33}$$

and for $J^2 = \frac{1}{2}(XY + YX) + (\frac{1}{2}H)^2$ we have

$$[J^2, H] = 0, \quad [J^2, X] = 0, \quad [J^2, Y] = 0. \tag{34}$$

which result from the use of (33) in the following

$$
\begin{aligned}
[J^2, H] &= [XY + YX, H] \\
&= X[Y,H] + [X,H]Y + Y[X,H] + [Y,H]X \\
&= X(2Y) + (-2X)Y + Y(-2X) + (2Y)X \\
&= 0
\end{aligned} \tag{35}
$$

then

$$
\begin{aligned}
[J^2, X] &= \frac{1}{2}[XY + YX, X] + \frac{1}{4}[H^2.X] \\
&= \frac{1}{2}\{X[Y,X] + [X,X]Y + Y[X,X] + [Y,X]X\} \\
&\quad + \frac{1}{4}\{H[H,X] + [H,X]H\} \\
&= \frac{1}{2}\{X(-H) + (-H)X\} + \frac{1}{4}\{H(2X) + (2X)H\} \\
&= 0
\end{aligned} \tag{36}
$$

and then

$$
\begin{aligned}
[J^2, Y] &= \frac{1}{2}[XY + YX, Y] + \frac{1}{4}[H^2.Y] \\
&= \frac{1}{2}\{X[Y,Y] + [X,Y]Y + Y[X,Y] + [Y,Y]X\} \\
&\quad + \frac{1}{4}\{H[H,Y] + [H,Y]H\} \\
&= \frac{1}{2}\{(H)Y + Y(H)\} + \frac{1}{4}\{H(-2Y) + (-2Y)H\} \\
&= 0.
\end{aligned} \tag{37}
$$

Corresponding analogous to H, X, Y and J^2 are:

$$
\begin{aligned}
&\tilde{\varphi}(0,0,-i) \equiv \tilde{\varphi}(0 \times u_1 + 0 \times u_2 - iu_3), \\
&\tilde{\varphi}(-\frac{i}{2}, \frac{1}{2}, 0) \equiv \tilde{\varphi}(-\frac{i}{2}u_1 + \frac{1}{2}u_2 + 0 \times u_3), \\
&\tilde{\varphi}(-\frac{i}{2}, -\frac{1}{2}, 0) \equiv \tilde{\varphi}(-\frac{i}{2}u_1 - \frac{1}{2}u_2 + 0 \times u_3), \\
&\frac{1}{2}(\tilde{\varphi}(-\frac{i}{2}, \frac{1}{2}, 0)\tilde{\varphi}(-\frac{i}{2}, -\frac{1}{2}, 0) + \tilde{\varphi}(-\frac{i}{2}, -\frac{1}{2}, 0)\tilde{\varphi}(-\frac{i}{2}, \frac{1}{2}, 0) + \frac{1}{2}\tilde{\varphi}(0,0,-i)\tilde{\varphi}(0,0,-i))
\end{aligned} \tag{38}
$$

respectively. Let us denote the last of these operations $\varphi(J^2)$.

Establishing analogues of (33)

We have from (6)

$$\begin{aligned}
\frac{1}{2}\tilde{\varphi}(-iu_3) &\quad . \quad (a_0 + a_1x + a_2x^2 + a_3y + a_4y^2 + a_5xy) \\
&= -\frac{1}{2}xa_1 + x^2a_2 + \frac{1}{2}ya_3 + y^2(-a_4) + \frac{1}{2}xy(0) \\
&= -\frac{1}{2}a_1x + \frac{1}{2}a_3y + a_2x^2 - a_4y^2. \qquad (39)
\end{aligned}$$

and

$$\begin{aligned}
\tilde{\varphi}(-\frac{i}{2}u_1 + \frac{1}{2}u_2) &\quad . \quad (a_0 + a_1x + a_2x^2 + a_3y + a_4y^2 + a_5xy) \\
&= x(-\frac{i}{2}a_3 + i\frac{1}{2}a_3) + x^2(2\frac{i}{2}a_5 + 2i\frac{1}{2}a_5) + y(\frac{i}{2}a_1 + i\frac{1}{2}a_1) + \\
&+ y^2(2\frac{i}{2}a_5 - 2i\frac{1}{2}a_5) + xy(-\frac{i}{2}a_2 - \frac{i}{2}a_4 + i\frac{1}{2}a_2 - i\frac{1}{2}a_4) \\
&= ia_1y + 2ia_5x^2 - ia_4xy. \qquad (40)
\end{aligned}$$

and

$$\begin{aligned}
\tilde{\varphi}(-\frac{i}{2}u_1 - \frac{1}{2}u_2) &\quad . \quad (a_0 + a_1x + a_2x^2 + a_3y + a_4y^2 + a_5xy) \\
&= x(-\frac{i}{2}a_3 - i\frac{1}{2}a_3) + x^2(2\frac{i}{2}a_5 - 2i\frac{1}{2}a_5) + y(\frac{i}{2}a_1 - i\frac{1}{2}a_1) + \\
&+ y^2(2\frac{i}{2}a_5 + 2i\frac{1}{2}a_5) + xy(-\frac{i}{2}a_2 - \frac{i}{2}a_4 - i\frac{1}{2}a_2 + i\frac{1}{2}a_4) \\
&= -ixa_3 + 2ia_5y^2 - ia_2xy. \qquad (41)
\end{aligned}$$

Using (39) and (40) we compute:

$$\begin{aligned}
&\frac{1}{2}\tilde{\varphi}(0,0,-i)\tilde{\varphi}(-\frac{i}{2},\frac{1}{2},0).(a_0 + a_1x + a_2x^2 + a_3y + a_4y^2 + a_5xy) = \\
&= \frac{1}{2}\tilde{\varphi}(0,0,-i).(ia_1y + 2ia_5x^2 - ia_4xy) \\
&= \frac{i}{2}a_1y + 2ia_5x^2
\end{aligned}$$

and

$$\begin{aligned}
&\tilde{\varphi}(-\frac{i}{2},\frac{1}{2},0)\frac{1}{2}\tilde{\varphi}(0,0,-i).(a_0+a_1x+a_2x^2+a_3y+a_4y^2+a_5xy)=\\
&=\tilde{\varphi}(-\frac{i}{2},\frac{1}{2},0).(-\frac{1}{2}a_1x+\frac{1}{2}a_3y+a_2x^2-a_4y^2)\\
&=-\frac{i}{2}a_1y+ia_4xy
\end{aligned}$$

from which we have

$$\begin{aligned}
&[\frac{1}{2}\tilde{\varphi}(0,0,-i),\tilde{\varphi}(-\frac{i}{2},\frac{1}{2},0)].(a_0+a_1x+a_2x^2+a_3y+a_4y^2+a_5xy)=\\
&=ia_1y+2ia_5x^2-ia_4xy\\
&=\tilde{\varphi}(-\frac{i}{2},\frac{1}{2},0).(a_0+a_1x+a_2x^2+a_3y+a_4y^2+a_5xy)
\end{aligned}\tag{42}$$

Using (39) and (41) we compute:

$$\begin{aligned}
&\frac{1}{2}\tilde{\varphi}(0,0,-i)\tilde{\varphi}(-\frac{i}{2},-\frac{1}{2},0).(a_0+a_1x+a_2x^2+a_3y+a_4y^2+a_5xy)=\\
&=\frac{1}{2}\tilde{\varphi}(0,0,-i).(-ixa_3+2ia_5y^2-ia_2xy)\\
&=\frac{i}{2}a_3x-2ia_5y^2
\end{aligned}$$

and

$$\begin{aligned}
&\tilde{\varphi}(-\frac{i}{2},-\frac{1}{2},0)\frac{1}{2}\tilde{\varphi}(0,0,-i).(a_0+a_1x+a_2x^2+a_3y+a_4y^2+a_5xy)=\\
&=\tilde{\varphi}(-\frac{i}{2},-\frac{1}{2},0).(-\frac{1}{2}a_1x+\frac{1}{2}a_3y+a_2x^2-a_4y^2)\\
&=-\frac{i}{2}a_3x-ia_2xy-ia_2xy
\end{aligned}$$

from which we have

$$\begin{aligned}
&[\frac{1}{2}\tilde{\varphi}(0,0,-i),\tilde{\varphi}(-\frac{i}{2},-\frac{1}{2},0)].(a_0+a_1x+a_2x^2+a_3y+a_4y^2+a_5xy)=\\
&=-(-ixa_3+2ia_5y^2-ia_2xy)\\
&=-\tilde{\varphi}(-\frac{i}{2},-\frac{1}{2},0).(a_0+a_1x+a_2x^2+a_3y+a_4y^2+a_5xy)
\end{aligned}\tag{43}$$

Using (40) and (41) we compute:

$$\begin{aligned}
&\tilde{\varphi}(-\frac{i}{2},\frac{1}{2},0)\tilde{\varphi}(-\frac{i}{2},-\frac{1}{2},0).(a_0+a_1x+a_2x^2+a_3y+a_4y^2+a_5xy)= \\
&=\tilde{\varphi}(-\frac{i}{2},\frac{1}{2},0).(-ixa_3+2ia_5y^2-ia_2xy) \\
&=a_3y+2a_5xy+2a_2x^2
\end{aligned}$$

and

$$\begin{aligned}
&\tilde{\varphi}(-\frac{i}{2},-\frac{1}{2},0)\tilde{\varphi}(-\frac{i}{2},\frac{1}{2},0).(a_0+a_1x+a_2x^2+a_3y+a_4y^2+a_5xy)= \\
&=\tilde{\varphi}(-\frac{i}{2},-\frac{1}{2},0).(ia_1y+2ia_5x^2-ia_4xy) \\
&=a_1x+2a_5xy+2a_4y^2
\end{aligned}$$

from which

$$\begin{aligned}
&[\tilde{\varphi}(-\frac{i}{2},\frac{1}{2},0),\tilde{\varphi}(-\frac{i}{2},-\frac{1}{2},0)].(a_0+a_1x+a_2x^2+a_3y+a_4y^2+a_5xy)= \\
&=-a_1x+a_3y+2a_2x^2-2a_4y^2 \\
&=\tilde{\varphi}(0,0,-i).(a_0+a_1x+a_2x^2+a_3y+a_4y^2+a_5xy).
\end{aligned}$$

Thus establishing the analogues of (33). Analogues of (34) obviously then follow.

Ladder operators, eigenvector, eiganvalues and all that

From

$$\tilde{\varphi}(J^2)\frac{1}{2}\tilde{\varphi}(0,0,-i)-\frac{1}{2}\tilde{\varphi}(0,0,-i)\tilde{\varphi}(J^2)=0 \tag{44}$$

we have that $\frac{1}{2}\tilde{\varphi}(0,0,-i)$ and $\tilde{\varphi}(J^2)$ have common (polynomial) eigenvectors.

From

$$\frac{1}{2}\tilde{\varphi}(0,0,-i)\tilde{\varphi}(-\frac{i}{2},\frac{1}{2},0)=\tilde{\varphi}(-\frac{i}{2},\frac{1}{2},0)(\frac{1}{2}\tilde{\varphi}(0,0,-i)+1) \tag{45}$$

we have that if a polynoimal $e(x,y)$ is an eigenvector of $\frac{1}{2}\tilde{\varphi}(0,0,-i)$ with eigenvalue p, then $\tilde{\varphi}(-\frac{i}{2},\frac{1}{2},0)e(x,y)$ is also an eigenvector of $\frac{1}{2}\tilde{\varphi}(0,0,-i)$ with eigenvalue $p+1$. Each time $\frac{1}{2}\tilde{\varphi}(0,0,-i)$ is applied the resulting eigenvector's eigenvalue will increase by one, until eventually it will annihilate the last eigenvector.

From

$$\frac{1}{2}\tilde{\varphi}(0,0,-i)\tilde{\varphi}(-\frac{i}{2},-\frac{1}{2},0)=\tilde{\varphi}(-\frac{i}{2},-\frac{1}{2},0)(\frac{1}{2}\tilde{\varphi}(0,0,-i)-1) \tag{46}$$

we have that if a polynoimal $e(x,y)$ is an eigenvector of $\frac{1}{2}\tilde{\varphi}(0,0,-i)$ with eigenvalue p, then $\tilde{\varphi}(-\frac{i}{2},-\frac{1}{2},0)e(x,y)$ is also an eigenvector of $\frac{1}{2}\tilde{\varphi}(0,0,-i)$ with eigenvalue $p-1$.

The $\varphi(\frac{1}{2},\frac{i}{2},0)$ and $\tilde{\varphi}(\frac{1}{2},-\frac{i}{2},0)$ are the so-called ladder up and down operators respectively.

From

$$\tilde{\varphi}(J^2)\varphi(-\frac{i}{2},\frac{1}{2},0)-\tilde{\varphi}(-\frac{i}{2},\frac{1}{2},0)\tilde{\varphi}(J^2)=0 \tag{47}$$

and

$$\tilde{\varphi}(J^2)\tilde{\varphi}(-\frac{i}{2},-\frac{1}{2},0)-\tilde{\varphi}(-\frac{i}{2},-\frac{1}{2},0)\tilde{\varphi}(J^2)=0 \tag{48}$$

it follows if $e(x,y)$ is an eigenvector of $\tilde{\varphi}(J^2)$ then so is $\tilde{\varphi}(-\frac{i}{2},\frac{1}{2},0)e(x,y)$ and $\tilde{\varphi}(-\frac{i}{2},-\frac{1}{2},0)e(x,y)$ with the same eigenvalue that $e(x,y)$ has (unless either ladder operation annihilates $e(x,y)$).

The eigenvalue of

$$\tilde{\varphi}(J^2)$$

will be denoted by $j(j+1)$ and the eigenvalue of

$$\frac{1}{2}\tilde{\varphi}(0,0,-i)$$

denoted by m, i.e. these eigenvalues are the pair $(j(j+1),m)$. The vector of maximal weight is the eigenvector with maximum value of m, which is obviously the vector annihilated by the ladder up operator.

The case of $\{1\}$

Consider the case $\{1\}$. We have from (41) and (40)

$$\begin{aligned} \tilde{\varphi}(-\frac{i}{2},-\frac{1}{2},0)\ .\ 1 &= 0 \\ \tilde{\varphi}(-\frac{i}{2},\frac{1}{2},0)\ .\ 1 &= 0. \end{aligned} \tag{49}$$

Then from (39)

$$\frac{1}{2}\tilde{\varphi}(0,0,-i)\ .\ 1 = 0 \tag{50}$$

and from (39), (40) and (41)

$$\begin{aligned} & \frac{1}{2}(\tilde{\varphi}(-\frac{i}{2},\frac{1}{2},0)\tilde{\varphi}(-\frac{i}{2},-\frac{1}{2},0)+\tilde{\varphi}(-\frac{i}{2},-\frac{1}{2},0)\tilde{\varphi}(-\frac{i}{2},\frac{1}{2},0)+\frac{1}{2}\tilde{\varphi}(0,0,-i)\tilde{\varphi}(0,0,-i))\ .\ 1 \\ = \ & 0. \end{aligned} \tag{51}$$

From (49) we easily see that the vector of maximal weight is 1, with the eigenvalues $(0,0)$.

The case of $\{x,y\}$

Consider the case $\{x,y\}$. We have from (41) and (40)

$$\begin{aligned} \tilde{\varphi}(-\frac{i}{2},-\frac{1}{2},0)\ .\ a_1x &= 0 \\ \tilde{\varphi}(-\frac{i}{2},\frac{1}{2},0)\ .\ a_1x &= ia_1y \\ \tilde{\varphi}(-\frac{i}{2},\frac{1}{2},0)\ .\ a_3y &= 0. \end{aligned} \tag{52}$$

Then from (39)

$$\frac{1}{2}\tilde{\varphi}(0,0,-i)\ .\ a_3y = \frac{1}{2}a_3y. \tag{53}$$

and from (39), (40) and (41)

$$
\begin{aligned}
& \frac{1}{2}(\tilde{\varphi}(-\frac{i}{2},\frac{1}{2},0)\tilde{\varphi}(-\frac{i}{2},-\frac{1}{2},0)+\tilde{\varphi}(-\frac{i}{2},-\frac{1}{2},0)\tilde{\varphi}(-\frac{i}{2},\frac{1}{2},0)+\frac{1}{2}\tilde{\varphi}(0,0,-i)\tilde{\varphi}(0,0,-i))\ .\ a_3y \\
= \ & \frac{1}{2}a_3(\tilde{\varphi}(-\frac{i}{2},\frac{1}{2},0)(-ix)+\frac{1}{2}y) \\
= \ & \frac{1}{2}a_3((-i)(iy)+\frac{1}{2}y) \\
= \ & \frac{3}{4}a_3y. \qquad (54)
\end{aligned}
$$

From (52) we easily see that the eigenvector of maximal weight is y, with the eigenvalues $(\frac{3}{4},\frac{1}{2})$.

The case $\{y^2, xy, x^2\}$

Consider the case $\{y^2, xy, x^2\}$. We have from (41) and (40)

$$
\begin{aligned}
\tilde{\varphi}(-\frac{i}{2},-\frac{1}{2},0)\ .\ a_4y^2 &= 0 \\
\tilde{\varphi}(-\frac{i}{2},\frac{1}{2},0)\ .\ a_4y^2 &= -ia_4xy \\
\tilde{\varphi}(-\frac{i}{2},\frac{1}{2},0)\ .\ a_5xy &= 2ia_5x^2 \\
\tilde{\varphi}(-\frac{i}{2},\frac{1}{2},0)\ .\ a_2x^2 &= 0. \qquad (55)
\end{aligned}
$$

Then from (39)

$$
\frac{1}{2}\tilde{\varphi}(0,0,-i)\ .\ a_2x^2 = a_2x^2 \qquad (56)
$$

and from (39), (40) and (41)

$$
\begin{aligned}
& \frac{1}{2}(\tilde{\varphi}(-\frac{i}{2},\frac{1}{2},0)\tilde{\varphi}(-\frac{i}{2},-\frac{1}{2},0)+\tilde{\varphi}(-\frac{i}{2},-\frac{1}{2},0)\tilde{\varphi}(-\frac{i}{2},\frac{1}{2},0)+\frac{1}{2}\tilde{\varphi}(0,0,-i)\tilde{\varphi}(0,0,-i))\ .\ a_2x^2 \\
= \ & \frac{1}{2}a_2(\tilde{\varphi}(-\frac{i}{2},\frac{1}{2},0)(-ixy)+2x^2) \\
= \ & \frac{1}{2}a_2((-i)(2ix^2)+2x^2) \\
= \ & 2a_2x^2. \qquad (57)
\end{aligned}
$$

From (55) we easily see that the vector of maximal weight is x^2, with the eigenvalues $(2,1)$.