

Problem on time derivative of a wave function in Hilbert space

Consider the anti-hermitian operator $\langle \hat{A}f|g \rangle = -\langle f|\hat{A}g \rangle$, f, g is the function in Hilbert space with the usual inner product $\langle f|g \rangle = \int f^* g dV$ and \hat{A} is a linear operator on Hilbert space

Proposition: if an operator \hat{A} which has the property $\langle \hat{A}f|f \rangle = -\langle f|\hat{A}f \rangle$ for all f in Hilbert space then \hat{A} is an anti-hermitian operator

Proof: consider $h = f + g$, f, g are in Hilbert space then $\langle \hat{A}h|h \rangle = -\langle h|\hat{A}h \rangle$

$$\rightarrow \langle \hat{A}f|f \rangle + \langle \hat{A}f|g \rangle + \langle \hat{A}g|f \rangle + \langle \hat{A}g|g \rangle = -(\langle f|\hat{A}f \rangle + \langle f|\hat{A}g \rangle + \langle g|\hat{A}f \rangle + \langle g|\hat{A}g \rangle)$$

$$\rightarrow \langle \hat{A}f|g \rangle + \langle \hat{A}g|f \rangle = -(\langle f|\hat{A}g \rangle + \langle g|\hat{A}f \rangle) \quad (1)$$

Since (1) is true for any f, g (in Hilbert space) then it's also true for ig (i is an imaginary number), replace g by ig then (1) becomes

$$\rightarrow \langle \hat{A}f|ig \rangle - \langle \hat{A}g|f \rangle = -(\langle f|\hat{A}ig \rangle - \langle ig|\hat{A}f \rangle) \quad (2)$$

From (1) and (2) $\langle \hat{A}f|g \rangle = -\langle f|\hat{A}g \rangle \rightarrow \hat{A}$ is an anti-hermitian operator

Problem: consider a wave function in quantum mechanics because it's the probability amplitude so $\langle \psi|\psi \rangle = 1$ (3) for all the wave equation and the wave equation is in Hilbert space so any operator in Hilbert space so the proposition still applied well

Therefore, consider the time derivative of (3) $\left\langle \frac{\partial \psi}{\partial t} \middle| \psi \right\rangle + \left\langle \psi \middle| \frac{\partial \psi}{\partial t} \right\rangle = 0$ or $\langle \partial_t \psi | \psi \rangle = -\langle \psi | \partial_t \psi \rangle$, and according to the previous proposition the time derivative operator is anti-hermitian, now let's consider the arbitrary operator \hat{A} which present some physical quantities then

$$\frac{d\langle A \rangle}{dt} = \frac{d}{dt} \langle \psi | \hat{A} \psi \rangle = \langle \partial_t \psi | \hat{A} \psi \rangle + \langle \psi | \partial_t (\hat{A} \psi) \rangle = -\langle \psi | \partial_t (\hat{A} \psi) \rangle + \langle \psi | \partial_t (\hat{A} \psi) \rangle = 0$$

Every physical quantity is conserved which is wrong, only energy and momentum should be conserved in all situation not all physical quantity