School of Mathematics and Statistics, University of NSW

MATH2501 Linear Algebra

PROBLEM SHEET 1

Linear Equations and Matrices

1. For each of the following matrices A and vectors $\mathbf{b}$ find the general solution of the system $A\mathbf{x} = \mathbf{b}$ by Gaussian elimination and also by use of the row-reduced echelon form.

(a) $A = \begin{pmatrix} 1 & -4 & -1 \\ -1 & 3 & 2 \\ 2 & -9 & 2 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 12 \\ -7 \\ 35 \end{pmatrix}$;

(b) $A = \begin{pmatrix} 0 & -3 & 1 \\ -1 & -1 & 2 \\ 2 & -7 & 1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} -1 \\ 0 \\ 7 \end{pmatrix}$;

(c) $A = \begin{pmatrix} 1 & -3 & 4 & -1 \\ 3 & -1 & 0 & 5 \\ -1 & 3 & -2 & 5 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 11 \\ 1 \\ 1 \end{pmatrix}$;

(d) $A = \begin{pmatrix} 3 & 4 & 2 & 0 & -3 \\ -1 & 0 & -2 & 4 & 1 \\ 2 & 7 & -3 & -5 & -2 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 3 \\ 7 \\ -26 \end{pmatrix}$.

2. Write down the general solutions for the following systems of equations

(a) $x_1 + 2x_2 = 4$; $x_2 + 3x_3 = 7$

(b) $x_1 + 2x_2 = 4$; $x_3 = 7$

(c) $x_1 + 2x_2 + x_4 = 5$; $x_3 - 5x_4 = 8$

(d) $x_1 + 2x_2 + 3x_3 + 4x_4 = 5$; $2x_1 + 4x_2 + 7x_3 + 6x_4 = 9$.

3. Show that the three equations $x + y + 2z = a$, $x + z = b$, $2x + y + 3z = c$ are consistent if $c = a + b$.

4. For which values of λ do the equations

$$\begin{aligned} x + 2y + \lambda z &= 1 \\ -x + \lambda y - z &= 0 \\ \lambda x - 4y + \lambda z &= -1 \end{aligned}$$

have

(a) no solutions

(b) infinitely many solutions

(c) a unique solution?

Find the possible solutions when they exist.

5. Find conditions on b_1, b_2, b_3 such that the triple of equations:

$$\begin{aligned} 2x_1 + 3x_2 + 4x_3 &= b_1, \\ 3x_1 + 4x_2 + 5x_3 &= b_2, \\ 4x_1 + 5x_2 + 6x_3 &= b_3 \end{aligned}$$

has a solution. Find a formula for this solution.

6. For each of the following matrices, find conditions (if any) on $b_1, b_2, \ldots$ such that there are solutions to $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b} = (b_1, \ldots, b_n)^T$.

(a) $A = \begin{pmatrix} 1 & -3 & 3 \\ 2 & -5 & 4 \\ 2 & -9 & 12 \end{pmatrix}$

(b) $A = \begin{pmatrix} 1 & -3 & 2 \\ -3 & 14 & -8 \\ -1 & -7 & 3 \end{pmatrix}$

(c) $A = \begin{pmatrix} -1 & 1 & 3 \\ 3 & 3 & -6 \\ -1 & 3 & -1 \\ 0 & -2 & 1 \end{pmatrix}$.

7. Let $A = \begin{pmatrix} 2 & -3 & 4 \\ 3 & 2 & -2 \\ 1 & -1 & 3 \end{pmatrix}$, $B = \begin{pmatrix} -2 & 1 \\ 3 & 4 \\ -1 & 5 \end{pmatrix}$, $C = \begin{pmatrix} -3 & 2 \\ 1 & -4 \\ 6 & 2 \end{pmatrix}$, $D = \begin{pmatrix} 2 & 3 & 1 \\ 1 & -2 & 3 \end{pmatrix}$. Evaluate each of the following that exists: $3A$, $-2B$, $A + B$, $B + C$, $A + 3I$, $B + 3I$, FD, DB, AB, BC, A^2, $(BD)^2$, A^T, B^T, B^TB, BB^T. Note I denotes the 3×3 identity matrix.

8. Use the standard formula to write down the inverse of the matrix $A = \begin{pmatrix} 4 & 5 \\ 3 & 4 \end{pmatrix}$.

9. Using the standard row-reduction algorithm, invert (if possible) the following matrices:

$$A = \begin{pmatrix} 0 & 7/2 & -1 \\ 2 & 1 & -4 \\ 1/2 & -3 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & -3 & -2 \\ 3 & -4 & -2 \\ -4 & 3 & 3 \end{pmatrix},$$

$$C = \begin{pmatrix} 1 & 3 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}, \ D = \begin{pmatrix} 0 & 2 & 0 \\ 1 & 2 & 3 \\ -1 & 4 & -2 \end{pmatrix}, \ E = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{pmatrix},$$

$$F = \begin{pmatrix} 1 & 4 & 1 \\ 2 & 3 & 1 \\ 1 & -7 & -2 \end{pmatrix}, \quad G = \begin{pmatrix} 5 & -6 & 1 & 5 \\ -3 & 5 & -1 & 4 \\ 2 & -2 & 1 & 2 \\ 1 & -1 & 0 & 1 \end{pmatrix}.$$

10. Show that if B is a square matrix then

(a) BB^T and $B + B^T$ are symmetric; and

(b) $B - B^T$ is skew-symmetric. (A matrix M is *skew-symmetric* if $M^T = -M$.)

Show that every square matrix is the sum of a symmetric matrix and a skew-symmetric matrix.

11. Find a 2×2 matrix A such that $A^2 = I$, but $A \neq \pm I$.

12. If K is a skew-symmetric matrix and $I + K$ is non-singular, prove that the matrix $(I + K)^{-1}(I - K)$ is orthogonal. (A matrix Q is called orthogonal if $Q^TD = I$.)

13. Given that A, B, C are $n \times n$ invertible matrices, A is symmetric, B is skew-symmetric and C is orthogonal simplify the following expressions.
(a) $A^{-1}(CB^2A)^TC$;
(b) $A(BA)^TB^{-1}C^6(BC^7)^TB$
(c) $C(BC)^{-1}B^TA^5(BA^4)^{-1}C(ABC)^T$.

14. Factorise each of the following matrices into the form PLU.

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \; B = \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix}, \; C = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 5 & 7 & 31 \end{pmatrix}$$

$$D = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 5 & 8 \\ 3 & 7 & 9 \end{pmatrix}, \quad E = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 5 & 9 & 6 \end{pmatrix}$$

$$F = \begin{pmatrix} 1 & 1 & 1 \\ 3 & 0 & 4 \\ 2 & 1 & 1 \end{pmatrix}, \quad G = \begin{pmatrix} 0 & 0 & 3 \\ 3 & 7 & 9 \\ 0 & 5 & 8 \end{pmatrix}.$$

Answers: **1.** (a) $(2,-3,2)^T$, (b) $(8,2,5)^T$, (c) $(2,5,6,0)^T + \lambda(-3,-4,-2,1)^T$, (d) $(5+\lambda-2\mu, -3+\mu, \mu, 3, \lambda)^T$. **2.** (a) $(-10,7,0)^T + s(6,-3,1)^T$, (b) $(4.0,7)^T + s(-2,1,0)^T$, (c) $(5,0,8,0)^T + s(-1,0,5,1)^T + t(-2,1,0,0)^T$ (d) $(8,0,-1,0)^T + s(-10,0,2,1)^T + t(-2,1,0,0)^T$. **4.** (a) $\lambda = \pm 2$. (b) $\lambda = 1$ (c) $\lambda \neq \pm 2, 1$. **5.** $b_1 = 2b_2 - b_3$. **6.** (a) $b_3 = 8b_1 - 3b_2$ (b) no conditions (c) $b_1 + b_2 + 2b_3 + 5b_4 = 0$. **7.** $A + B$ undefined; $B + C = \begin{pmatrix} -5 & 3 \\ 4 & 0 \\ 5 & 7 \end{pmatrix}$; $A + 3I = \begin{pmatrix} 5 & -3 & 4 \\ 3 & 5 & -2 \\ 1 & -1 & 6 \end{pmatrix}$; $B + 3I$ undefined; $BD = \begin{pmatrix} -3 & -8 & 1 \\ 10 & 1 & 15 \\ 3 & -13 & 14 \end{pmatrix}$; $DB = \begin{pmatrix} 4 & 19 \\ -11 & 8 \end{pmatrix}$; $AB = \begin{pmatrix} -17 & 10 \\ 2 & 1 \\ -8 & 12 \end{pmatrix}$; BC undefined; $A^2 = \begin{pmatrix} -1 & -16 & 26 \\ 10 & -3 & 2 \\ 2 & -8 & 15 \end{pmatrix}$; $(BD)^2 = \begin{pmatrix} -68 & 3 & -109 \\ 25 & -274 & 235 \\ -97 & -219 & 4 \end{pmatrix}$; $A^T = \begin{pmatrix} 2 & 3 & 1 \\ -3 & 2 & -1 \\ 4 & -2 & 3 \end{pmatrix}$; $B^T = \begin{pmatrix} -2 & 3 & -1 \\ 1 & 4 & 5 \end{pmatrix}$; $BB^T = \begin{pmatrix} 5 & -2 & 7 \\ -2 & 25 & 17 \\ 7 & 17 & 26 \end{pmatrix}$; $B^TB = \begin{pmatrix} 14 & 5 \\ 5 & 42 \end{pmatrix}$. **8.** $\begin{pmatrix} 4 & -5 \\ -3 & 4 \end{pmatrix}$. **9.** $A^{-1} = \begin{pmatrix} 24 & -6 & 26 \\ 4 & -1 & 4 \\ 13 & -7/2 & 14 \end{pmatrix}$ $B^{-1} = \begin{pmatrix} 6 & -3 & 2 \\ 1 & -1 & 0 \\ 7 & -3 & 3 \end{pmatrix}$ $C^{-1} = \begin{pmatrix} 1 & -3 & -4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$ $D^{-1} = \begin{pmatrix} 8 & -2 & -3 \\ 1/2 & 0 & 0 \\ -3 & 1 & 1 \end{pmatrix}$ E has no inverse. $F^{-1} = \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 \\ 5 & -3 & 1 \\ -17 & 11 & -5 \end{pmatrix}$. **11.** Many examples exist e.g. $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ **13.** (a) B^2 (b) $A^2C^TB^2$ (c) A^2. **14.** Many solutions are possible. Check your answers by multiplication.

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PROBLEM SHEET 2

Vector Spaces Reviewed

1. Are the following sets subspaces of $\mathbb{R}^3$? Give reasons for your answers:
(a) $\{\mathbf{x} \mid x_1^2 = x_2^3\}$;
(b) $\{\mathbf{x} \mid 2x_1 - 3x_2 + 3x_3 = 13\}$;
(c) $\{\mathbf{x} \mid 2x_1 - 3x_2 + 3x_3 = 0\}$;
(d) $\{\mathbf{x} \mid \mathbf{x} = t_1\mathbf{u_1} + t_2\mathbf{u_2}, t_1, t_2 \in \mathbb{R}\}$ ($\mathbf{u_1}$ and $\mathbf{u_2}$ are fixed elements of $\mathbb{R}^3$);
(e) $\{\mathbf{x} \mid 2x_1 + 3x_2 - 4x_3 = 6\}$;
(f) $\{\mathbf{x} \mid 2x_1 + 3x_2 - 4x_3 = 0\}$.
Give a basis for those sets that are subspaces.

2. Is the set

$$\mathcal{S} = \{p \in \mathcal{P}_3 \mid xp'(x) - 2p(x) = 0 \text{ for all } x \in \mathbb{R}\}$$

a subspace of $\mathcal{P}_3$, the space of all polynomial functions of x of degree ≤ 3?

3. Let $\mathcal{V} = C[-1, 1]$ be the vector space of real valued continuous functions on $[-1, 1]$, where addition is defined by $(f + g)(t) := f(t) + g(t)$ and scalar multiplication is defined by $(\lambda f)(t) = \lambda f(t)$ for all $f, g \in \mathcal{V}$ and $\lambda \in \mathbb{R}$. Define

$$\mathcal{U} := \left\{ f \in \mathcal{V} \mid \int_{-1}^{1} f(t)dt = f(0) \right\} \quad \text{and} \quad \mathcal{W} := \left\{ f \in \mathcal{V} \mid \int_{-1}^{1} f(t)dt = 1 \right\}.$$

Prove that $\mathcal{U}$ is a vector space, but $\mathcal{W}$ is not.

4. Let $\mathcal{V}$ be the set of all sequences $\{\alpha_n\} = \{\alpha_0, \alpha_1, \ldots, \alpha_n, \ldots\}$ of elements of $\mathbb{R}$. Define the addition of sequences by $\{\alpha_n\} + \{\beta_n\} := \{\alpha_n + \beta_n\}$ and scalar multiplication by $\lambda\{\alpha_n\} := \{\lambda\alpha_n\}$. Prove that $\mathcal{V}$ is a vector space over $\mathbb{R}$.

5. ($*$) Prove that the following are vector spaces over $\mathbb{R}$:
(a) The set $\{\{\alpha_n\} \mid \alpha_{n+1} - \alpha_n = \alpha_{n+2} - \alpha_{n+1}, n \geq 0, \alpha_n \in \mathbb{R}\}$ of all arithmetic progressions;
(b) The set $\{\{\alpha_n\} \mid \alpha_{n+2} = \alpha_{n+1} + \alpha_n, n \geq 0, \alpha_n \in \mathbb{R}\}$ of all Fibonacci sequences;
(c) The set $\{\{\alpha_n\} \mid \{\alpha_n\} \text{ converges }, \alpha_n \in \mathbb{R}\}$ of all convergent sequences.

6. Are the vectors $\mathbf{v}_1 = (1, 2, 3)^T$, $\mathbf{v}_2 = (2, 3, 4)^T$ and $\mathbf{v}_3 = (3, 4, 5)^T$ in $\mathbb{R}^3$ independent?

7. The set $S = \{\mathbf{u}, \mathbf{v}\}$ is a linearly independent set in a real vector space $\mathcal{V}$. Suppose that $\mathbf{w} \in \mathcal{V}$ is such that $\mathbf{w} \notin \text{span}(S)$. Show that $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a linearly independent set.

8. Is the polynomial $p(x) = 1 + x + x^2$ in $\text{span}\{1 - x + 2x^2, -1 + x^2, -2 - x - 5x^2\}$?

9. Show that the set $\{1, e^x, e^{2x}, e^{3x}\}$ is linearly independent in the vector space $\mathcal{F}$ of all real-valued functions on $\mathbb{R}$.

10. Consider a possible identity $\lambda_1 \cos t + \lambda_2 \sin t + \lambda_3 t \cos t + \lambda_4 t \sin t = 0$ for all t. By evaluating this identity at certain values of t extract a system of linear equations for the λ_i and hence show that the four functions $\cos t, \sin t, t\cos t$ and $t \sin t$ are linearly independent.

11. Set $f(t) = \lambda_1 \cos t + \lambda_2 \sin t + \lambda_3 \cos 2t + \lambda_4 \sin 2t$. Evaluate

$$\int_0^{2\pi} f(t) \cos t\, dt, \quad \int_0^{2\pi} f(t) \sin t\, dt,$$

$$\int_0^{2\pi} f(t) \cos 2t\, dt \quad \text{and} \quad \int_0^{2\pi} f(t) \sin 2t\, dt$$

to show that the four functions $\cos t, \sin t, \cos 2t, \sin 2t$ are linearly independent.

12. Are the four cubics

$$p_1 = -1 - 3x^2 + x^3, \quad p_2 = -1 + 2x + x^2,$$
$$p_3 = 1 - 2x + x^3, \quad p_4 = 1 - 6x - 4x^2 + 3x^3$$

linearly dependent in $\mathcal{P}_3(\mathbb{R})$? If so, write p_4 as a linear combination of the other three. Otherwise, write $p = -2 + 6x + 2x^2 - x^3$ as a linear combination of all four cubics.

13. Let $\mathcal{V}$ be the vector space of all twice-differentiable real-valued functions defined on $\mathbb{R}$. If $f, g \in \mathcal{V}$, then define

$$W_{f,g}(t) := \begin{vmatrix} f(t) & g(t) \\ f'(t) & g'(t) \end{vmatrix}.$$

(a) Prove that if $W_{f,g}(t) \neq 0$ then $\{f, g\}$ is a linearly independent set in $\mathcal{V}$.
(b) Show that $\{\sin t, \cos t\}$ is a linearly independent set.
In general this *Wronskian determinant* is significant in the theory of linear ordinary differential equations.

14. Find the co-ordinate vector of $(b_1, b_2)^T \in \mathbb{R}^2$ with respect to the basis $B = \{(2, 3)^T, (3, 5)^T\}$.

15. Show that the columns of the matrix A below are not a spanning set for $\mathbb{R}^4$. Find a basis for $\mathbb{R}^4$ which contains as many of the columns of A as possible.

$$A = \begin{pmatrix} 1 & 3 & 3 & -7 & 5 \\ 2 & 6 & 3 & -8 & 1 \\ 3 & 9 & 2 & -7 & 3 \\ 4 & 12 & 0 & -4 & 11 \end{pmatrix}.$$

16. Let $\mathcal{V} = \mathcal{P}_3(\mathbb{R})$ be the vector space of polynomials with real coefficients and degree at most 3. Show that the set

$$\{t^2 + t^3, 1 + t + t^2 + t^3, t, 1 + t^2 + 2t^3\}$$

is linearly independent in $\mathcal{V}$. Explain why this set forms a basis for $\mathcal{V}$ and obtain the coordinates of $1 + 2t + 3t^2 + 4t^3$ with respect to this basis.

17. Find a basis for the column space and the row space of the matrix $\begin{pmatrix} 3 & 4 & 5 \\ 4 & 5 & 6 \\ 5 & 6 & 7 \end{pmatrix}$

18. Use row-reduction to find a basis for the column space of A and a basis for the row space of B where A and B are the following matrices:

$$A = \begin{pmatrix} -2 & 1 \\ 3 & 4 \\ -1 & 5 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 3 & 1 \\ 1 & -2 & -3 \end{pmatrix}.$$

19. Find bases for the kernels (nullspaces, $\mathcal{NS}$), row spaces ($\mathcal{RS}$) and column spaces ($\mathcal{CS}$) of each of the following matrices. Hence obtain their nullities and ranks.

$$A = \begin{pmatrix} 1 & -2 & 5 & 3 \\ -3 & 5 & 3 & 2 \\ 3 & -5 & 5 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & -3 & -1 \\ 2 & -3 & -1 \\ -3 & 3 & 2 \end{pmatrix}$$

$$\text{and } C = \begin{pmatrix} 2 & 3 & -4 & 2 & -3 \\ 1 & 1 & 2 & 0 & 0 \\ -4 & 0 & -4 & 2 & -3 \end{pmatrix}.$$

Find the co-ordindates of the vector $\mathbf{v} = (6, 6, -8)^T$ with respect to your basis of $\mathcal{CS}(B)$ and with respect to your basis of $\mathcal{RS}(B)$.

20. For what values of the unknown k do the following matrices have non-trivial nullspaces? Find the nullspaces in these cases.

(a) $A = \begin{pmatrix} 1 & 1 - 2k \\ k - 3 & 3 \end{pmatrix}$

(b) $B = \begin{pmatrix} k & 2 & 2 \\ 3k & 7 + k & 7 \\ 1 & -2k - 2 & -3 \end{pmatrix}$.

Answers: **1.** (a) No (b) No, (c) Yes, (d) Yes, (e) No, (f) Yes. **2.** Yes. **6.** No. **8.** Yes. **12.** No. $p = p_1 + p_2 + p_3 - p_4$. **14.** $\begin{pmatrix} 5b_1 - 3b_2 \\ 2b_2 - 3b_1 \end{pmatrix}$ **15.** $\{(1, 2, 3, 4)^T, (3, 3, 2, 0)^T, (5, 1, 3, 11)^T, (1, 0, 0, 0)^T\}$ for example. **16.** $(2, 0, 2, 1)^T$. **17.** $\{(3, 4, 5)^T, (4, 5, 6)^T\}$. **18.** $CS(A) = \text{span}\{(-2, 3, -1)^T, (1, 4, 5)^T\}$ $RS(B) = \text{span}\{(1, 0, -1)^T, (0, 1, 1)^T\}$. **19.** $\text{Ker}(A) = \text{span}\{(7, 4, -1, 2)^T$, $CS(A) = \mathbb{R}^3$, $RS(A) = \text{span}\{(1, -2, 5, 3)^T, (0, -1, 18, 11)^T, (0, 0, 2, 1)^T\}$. $\text{Ker}(B) = \text{span}\{(3, 1, 3)^T\}$, $CS(B) = \text{span}\{(2, 2, -3)^T, (1, 1, -1)^T\}$, $RS(B) = \text{span}\{(2, -3, -1)^T, (0, -3, 1)^T\}$. $\text{Ker}(C) = \text{span}\{(2, -4, 1, 6, 0)^T, (0, 0, 0, 3, 2)^T\}$, $CS(C) = \mathbb{R}^3$, $RS(C) = \text{span}\{(2, 3, -4, 2, -3)^T, (0, 2, -4, 2, -3)^T, (0, 0, 12, -4, 3)^T\}$. $[2, 2]$ for basis of $CS(B)$ above and $[3, -5]$ for basis of $RS(B)$ above. **20.** (a) for $k = 2$, $\text{Ker}(A) = \text{span}\{(3, 1)^T\}$ and for $k = \frac{3}{2}$ $\text{Ker}(A) = \text{span}\{(2, 1)^T\}$ (b) for $k = 0$, $\text{Ker}(A) = \text{span}\{(1, -1, 1)^T\}$ and for $k = -3$, $\text{Ker}(A) = \text{span}\{(2, 1, 2)^T\}$

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MATH2501 Linear Algebra

PROBLEM SHEET 3

Linear Transformations

1. For each of the following functions either show that they are linear and find their matrices with respect to the standard bases, or prove they are not linear:

(a) $T:\mathbb{R}^2\to\mathbb{R}^2$, $T\begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}2a\\a-b\end{pmatrix}$.

(b) $T:\mathbb{R}^2\to\mathbb{R}^2$, $T\begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}a+1\\b\end{pmatrix}$.

(c) $T:\mathcal{P}_2(\mathbb{R})\to\mathcal{P}_2(\mathbb{R})$, $T(a+bt+ct^2)=(a+c)-(c+b)t+(a+b+c)t^2$.

(d) $T:\mathcal{P}_2(\mathbb{R})\to\mathcal{P}_2(\mathbb{R})$, $T(a+bt+ct^2)=a+b(t+1)+c(t+1)^2$.

(e) $f:\mathbb{R}^3\to\mathbb{R}^2$, $f(x_1,x_2,x_3)^T=(9x_1-2x_2,x_1-5x_2)^T$.

(f) $f:\mathbb{R}^3\to\mathbb{R}^2$, $f(x_1,x_2,x_3)^T=(9x_1-2x_2,x_1-5x_2-3)^T$.

2. Let $\mathcal{V}$ and $\mathcal{W}$ be vector spaces and $T:\mathcal{V}\to\mathcal{W}$ be linear. Show that T is uniquely determined by the values of $T(\mathbf{v}_i)$ for every member $\mathbf{v}_i$ of any basis of $\mathcal{V}$.

3. Given that each of the following maps f is linear and takes the given values on the given basis vectors for $\mathbb{R}^2$, find formulae for $f(x_2,x_2)^T$.

(a) $f(1,0)^T=(3,4)^T$, $f(0,1)^T=(4,9)^T$.

(b) $f(4,7)^T=(3,4)^T$, $f(3,5)^T=(4,9)^T$.

(c) $f(5,7)^T=(3,4)^T$, $f(2,7)^T=(2,5)^T$.

4. Let $\mathcal{P}_2(\mathbb{R})$ denote the space of polynomials with real coefficients and degree ≤ 2. The function f from $\mathcal{P}_2(\mathbb{R})$ to $\mathcal{P}_2(\mathbb{R})$ is defined by $f(p)(x)=p'(x)$.

(a) Show f is linear.

(b) Find the matrix of f with respect to the basis $B=\{1,1+x,x^2\}$

5. Let f,g be maps $\mathcal{P}_1(\mathbb{R})\to\mathbb{R}^2$, defined by $f(a+bx)=\begin{pmatrix}2a+3b\\a-b\end{pmatrix}$ and $g(a+bx)=\begin{pmatrix}a-b\\a+b\end{pmatrix}$.

(a) Show that f and g are linear maps.

(b) Find the matrices of f and g relative to the bases $S'=\{1,x\}$ in $\mathcal{P}_1(\mathbb{R})$ and $S=\left\{\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\right\}$ in $\mathbb{R}^2$.

(c) Find the matrices of f and g relative to the bases $\{1+x,\ 2-x\}$ in $\mathcal{P}_1(\mathbb{R})$ and $\left\{\begin{pmatrix}2\\3\end{pmatrix},\begin{pmatrix}1\\2\end{pmatrix}\right\}$ in $\mathbb{R}^2$.

6. (a) Show that $B=\left\{\begin{pmatrix}1\\5\end{pmatrix},\begin{pmatrix}1\\6\end{pmatrix}\right\}$ is a basis for $\mathbb{R}^2$.

(b) Suppose that $T:\mathbb{R}^2\to\mathbb{R}^2$ is linear and has matrix $\begin{pmatrix}-2&1\\5&2\end{pmatrix}$ with respect to the standard basis S of $\mathbb{R}^2$. What is the matrix of T with respect to B?

7. Let $B=\left\{\begin{pmatrix}1\\-1\end{pmatrix},\begin{pmatrix}-3\\2\end{pmatrix}\right\}$ be a basis for $\mathbb{R}^2$ and suppose that $T:\mathbb{R}^2\to\mathbb{R}^2$ is linear and has matrix $\begin{pmatrix}4&9\\1&1\end{pmatrix}$ with respect to the standard basis S of $\mathbb{R}^2$. What is the matrix of T with respect to B?

8. Let $B=\left\{\begin{pmatrix}1\\4\end{pmatrix},\begin{pmatrix}1\\3\end{pmatrix}\right\}$ be a basis for $\mathbb{R}^2$ and suppose that $T:\mathbb{R}^2\to\mathbb{R}^2$ is linear and has matrix $\begin{pmatrix}6&-1\\12&-1\end{pmatrix}$ with respect to the standard basis S of $\mathbb{R}^2$. What is the matrix of T with respect to B?

9. $\mathcal{V}$ is a 3-dimensional vector space over $\mathbb{R}$ with basis $B=\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$. Let f be a linear map from $\mathcal{V}$ to $\mathcal{V}$ such that $f(\mathbf{v}_1)=2\mathbf{v}_1+3\mathbf{v}_3$, $f(\mathbf{v}_2)=4\mathbf{v}_2+\mathbf{v}_3$, $f(\mathbf{v}_3)=-\mathbf{v}_1+2\mathbf{v}_2+6\mathbf{v}_3$. Write down the matrices of f, f^2 (that is, $f\circ f$), $2I-f$, with respect to the given basis. Here $I\colon\mathcal{V}\to\mathcal{V}$ denotes the identity map.

10. Let $\mathcal{P}_n(\mathbb{R})$ be the vector space of polynomials of degree at most n with coefficients in $\mathbb{R}$.

(a) What is the rank of the differentiation operator on $\mathcal{P}_3(\mathbb{R})$? What is its null space? What is its matrix with respect to the standard basis $\mathcal{S}$ of $\mathcal{P}_3(\mathbb{R})$ in both domain and codomain and its matrix with respect to the basis $\{1,1+t,1+t+t^2,1+t+t^2+t^3\}$ in both domain and codomain?

(b) Verify that the matrix of differentiation three times is the cube of the matrix of differentiation once on $\mathcal{P}_3(\mathbb{R})$.

11. Let $\mathcal{V}=\mathcal{P}_3(\mathbb{R})$ be the vector space of polynomials with real coefficients and degree at most 3.

(a) Show that the map $T\colon\mathcal{V}\to\mathbb{R}^2$ given by $T(\mathbf{v})=(\mathbf{v}'(0),\mathbf{v}(0))^T$ for all $\mathbf{v}\in\mathcal{V}$ is linear and obtain its matrix with respect to the standard basis S' for $\mathcal{V}$ in the domain and the basis S for $\mathbb{R}^2$ in the codomain.

(b) Find $\mathrm{Ker}(T)$ and $\dim(\mathrm{Ker}(T))$ and also the image of T, $\mathrm{im}(T)$, in other words find bases for these subspaces.

(c) Consider the bases $B'=\{1,1+t,t+t^2,t^2+t^3\}$ of $\mathcal{V}$ and $B=\{(1,1)^T,(1,-1)^T\}$ of $\mathbb{R}^2$. Find the matrices associated with the change of bases from these new bases to the standard bases in part **a** and then use those matrices to calculate the matrix of T with respect to the new bases.

Answers: **1.** (a) linear, $\begin{pmatrix}2 & 0\\ 1 & -1\end{pmatrix}$ (b) not linear‘ (c) linear $\begin{pmatrix}1 & 0 & 1\\ 0 & -1 & -1\\ 1 & 1 & 1\end{pmatrix}$ (d) linear $\begin{pmatrix}1 & 1 & 1\\ 0 & 1 & 2\\ 0 & 0 & 1\end{pmatrix}$ (e) linear, $\begin{pmatrix}9 & -2 & 0\\ 1 & -5 & 0\end{pmatrix}$. (f) not linear. **3.** (a) $f(x_1, x_2)^T = (3x_1 + 4x_2, 4x_1 + 9x_2)^T$. (b) $f(x_1, x_2)^T = (13x_1 - 7x_2, 43x_1 - 24x_2)^T$. **4.** $\begin{pmatrix}0 & 1 & -2\\ 0 & 0 & 2\\ 0 & 0 & 0\end{pmatrix}$. **5.** (b) $\begin{pmatrix}2 & 3\\ 1 & -1\end{pmatrix}$, $\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}$. (c) $\begin{pmatrix}10 & -1\\ -15 & 3\end{pmatrix}$, $\begin{pmatrix}-2 & 5\\ 4 & -7\end{pmatrix}$. **6.** $\begin{pmatrix}3 & 7\\ 0 & -3\end{pmatrix}$. **7.** $\begin{pmatrix}10 & -9\\ 5 & -5\end{pmatrix}$. **8.** $\begin{pmatrix}2 & 0\\ 0 & 3\end{pmatrix}$. **9.** $[f]_{BB} = \begin{pmatrix}2 & 0 & -1\\ 0 & 4 & 2\\ 3 & 1 & 6\end{pmatrix}$ $[f^2]_{BB} = \begin{pmatrix}1 & -1 & -8\\ 6 & 18 & 20\\ 24 & 10 & 35\end{pmatrix}$ $[2I - f]_{BB} = \begin{pmatrix}0 & 0 & 1\\ 0 & -2 & -2\\ -3 & -1 & -4\end{pmatrix}$ **10.** (a) Rank is 3, null space span$\{1\}$. Matrices $\begin{pmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 2 & 0\\ 0 & 0 & 0 & 3\\ 0 & 0 & 0 & 0\end{pmatrix}$ and $\begin{pmatrix}0 & 1 & -1 & -1\\ 0 & 0 & 2 & -1\\ 0 & 0 & 0 & 3\\ 0 & 0 & 0 & 0\end{pmatrix}$ **11.** (a) $\begin{pmatrix}0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\end{pmatrix}$. (b) $\mathrm{im}(T) = \mathbb{R}^2$, $\mathrm{Ker}(T) = \mathrm{span}\{t^2, t^3\}$ so has dimension 2. (c) Matrix for $\mathcal{V}$ is $\begin{pmatrix}1 & 1 & 0 & 0\\ 0 & 1 & 1 & 0\\ 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 1\end{pmatrix}$, for $\mathbb{R}^2$ is $\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$. Matrix of T is $\frac{1}{2}\begin{pmatrix}1 & 2 & 1 & 0\\ -1 & 0 & 1 & 0\end{pmatrix}$.

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MATH2501 Linear Algebra

PROBLEM SHEET 4

Least Squares, Gram-Schmidt.

1. Find the projection of $(x_1, x_2)^T \in \mathbb{R}^2$ on the line spanned by $(5, 12)^T$.

2. Find the projection in $\mathbb{R}^3$ (with the usual inner product) of $(x_1, x_2, x_3)^T$ on the subspace spanned by $(1, 2, 2)^T$ and $(2, 1, -2)^T$.

3. Find the projection in $\mathbb{R}^3$ (with the usual inner product) of $(x_1, x_2, x_3)^T$ on the subspace of $\mathbb{R}^3$ orthogonal to the subspace spanned by $(1, 2, 2)^T$ and $(2, 1, -2)^T$.

4. Let M be the matrix $\begin{pmatrix} -1 & 7 & 11 \\ 8 & 7 & 11 \\ 4 & 8 & 1 \end{pmatrix}$. Show that the columns of M form a basis for $\mathbb{R}^3$ and use the Gram-Schmidt process to modify this basis to obtain an orthonormal basis.

5. Now use your calculations in Q.4 above to find a QR factorisation of the matrix M.

6. Let $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be an orthonormal basis for the inner product space $\mathcal{V}$. Show that any $\mathbf{v} \in \mathcal{V}$ can be written $\mathbf{v} = \langle \mathbf{v}_1, \mathbf{v} \rangle \mathbf{v}_1 + \ldots + \langle \mathbf{v}_n, \mathbf{v} \rangle \mathbf{v}_n$.

7. In this question $\mathbf{x} = (x_1, x_2)^T$ and $\mathbf{y} = (y_1, y_2)^T$ denote two general elements of the vector space $\mathbb{R}^2$. Which of the following functions are inner products on $\mathbb{R}^2$? Give reasons.

(a) $\langle \mathbf{x}, \mathbf{y} \rangle = x_1x_2 + y_1y_2$;
(b) $\langle \mathbf{x}, \mathbf{y} \rangle = x_1y_2 + y_1x_2$;
(c) $\langle \mathbf{x}, \mathbf{y} \rangle = x_1y_1 + 3x_2y_2$;
(d) $\langle \mathbf{x}, \mathbf{y} \rangle = x_1y_2 + 2y_1x_2$;
(e) $\langle \mathbf{x}, \mathbf{y} \rangle = x_1y_1 - 3x_2y_2$?

8. For a real inner product space:

(a) Prove the identity $\|\mathbf{u}+\mathbf{v}\|^2 + \|\mathbf{u}-\mathbf{v}\|^2 = 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2$.
Why is this identity known as the *parallelogram identity*?
(b) Prove the identity $4(\mathbf{u}\cdot\mathbf{v}) = \|\mathbf{u}+\mathbf{v}\|^2 - \|\mathbf{u}-\mathbf{v}\|^2$.
(c) Prove *Pythagoras' Theorem*: $\mathbf{u}\cdot\mathbf{v} = 0$ if and only if $\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 = \|\mathbf{v}+\mathbf{u}\|^2$.

9. (**H**) Let $\mathbf{u}$ and $\mathbf{v}$ be two non-zero vectors in $\mathbb{R}^n$. Show that the vector $\mathbf{w} = (\|\mathbf{u}\| + \|\mathbf{v}\|)^{-1}(\|\mathbf{u}\|\mathbf{v} + \|\mathbf{v}\|\mathbf{u})$ bisects the angle between $\mathbf{u}$ and $\mathbf{v}$.

10. Use the Gram-Schmidt process to find orthonormal bases for the spaces:

(a) $\text{span}\{(5, 12)^T, (-4, 6)^T\}$;
(b) $\text{span}\{(-2, 1, -2)^T, (1, 4, -8)^T\}$;
(c) $\text{span}\{(2, -3, 6)^T, (1, 1, -1)^T\}$.

11. Apply the Gram-Schmidt process to the standard basis of $\mathcal{P}_3(\mathbb{R})$ to find an orthogonal basis for $\mathcal{P}_3(\mathbb{R})$ using as inner product

$$\langle p, q \rangle = \int_{-1}^{1} p(t)q(t)\, dt\ .$$

These polynomials are called *Legendre polynomials*.

12. (a) Let $\mathcal{C}[-\pi, \pi]$ be the set of real valued continuous functions on $[-\pi, \pi]$. If $\langle f, g \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)g(t)\, dt$ show that $\langle f, g \rangle$ defines an inner product on $C[-\pi, \pi]$.

(b) Show that the set

$$\{1/\sqrt{2}, \cos t, \sin t, \cos 2t, \sin 2t, \ldots\}$$

is an orthonormal set with respect to this inner product.

13. Let $\mathcal{V}$ be a finite-dimensional inner product space and let $\mathcal{W} \subseteq \mathcal{V}$ be a subspace.

Define $\mathcal{W}^\perp = \{\mathbf{v} \in \mathcal{V} \,|\, \langle \mathbf{v}, \mathbf{w} \rangle = 0 \text{ for all } \mathbf{w} \in \mathcal{W}\}$. Show that:

(a) $\mathcal{W}^\perp$ is a subspace of $\mathcal{V}$.
(b) $\mathcal{W} \cap \mathcal{W}^\perp = \{\mathbf{0}\}$.
(c) $\dim \mathcal{W} + \dim \mathcal{W}^\perp = \dim \mathcal{V}$.
(d) $\mathcal{W}^{\perp\perp} = \mathcal{W}$.
(e) If $\mathbf{v} \in \mathcal{V}$, there are unique vectors $\mathbf{w}_1 \in \mathcal{W}$, $\mathbf{w}_2 \in \mathcal{W}^\perp$ such that $\mathbf{v} = \mathbf{w}_1 + \mathbf{w}_2$.
(f) Assuming $\mathcal{V} = \mathbb{R}^n$, $p_{\mathcal{W}} + p_{\mathcal{W}^\perp} = I$, where $p_{\mathcal{W}}$ and $p_{\mathcal{W}^\perp}$ are the orthogonal projections on $\mathcal{W}$ and $\mathcal{W}^\perp$ respectively.

14. (a) For $\mathbb{R}^3$ with the usual inner product calculate $\text{span}\{(1, 1, 1)^T, (1, -1, 1)^T\}^\perp$.

(b) In $\mathbb{R}^4$ with the usual inner product find a vector that is orthogonal to $(1, 1, 1, -1)^T$, $(2, 1, 1, 1)^T$, and $(1, 2, 0, 1)^T$.

15. Let $\mathcal{W} = \text{Ker}\begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & 2 & -2 & 1 \end{pmatrix}$.

(a) Calculate $\mathcal{W}^\perp$.
(b) Find orthonormal bases of $\mathcal{W}$ and $\mathcal{W}^\perp$.
(c) For $\mathbf{v} = (1, -1, 2, 3)^T$, find $\mathbf{w}_1 \in \mathcal{W}$ and $\mathbf{w}_2 \in \mathcal{W}^\perp$ such that $\mathbf{v} = \mathbf{w}_1 + \mathbf{w}_2$.

16. For the following subspaces $\mathcal{W}$ find the matrix P of the projection onto $\mathcal{W}$ and the projection of the given vector $\mathbf{v}$ onto $\mathcal{W}$:

(a) $\mathcal{W} = \text{span}\{(-2, 1, -2)^T, (1, 4, -8)^T\}$, $\mathbf{v} = (0, 5, 0)^T$;
(b) $\mathcal{W} = \text{span}\{(1, -1, 1)^T, (-1, 2, -3)^T\}$, $\mathbf{v} = (1, 0, 5)^T$.

17. Show that
$$A = \begin{pmatrix} 5 & 9 \\ 12 & 11 \end{pmatrix} = \begin{pmatrix} 5/13 & 12/13 \\ 12/13 & -5/13 \end{pmatrix} \begin{pmatrix} 13 & 177/13 \\ 0 & 53/13 \end{pmatrix}$$
is a QR factorisation for the matrix A.

18. Find a QR factorisation for each of the matrices:
(a) $A = \begin{pmatrix} 5 & -4 \\ 12 & 6 \end{pmatrix}$;
(b) $B = \begin{pmatrix} -2 & 1 \\ 1 & 4 \\ -2 & -8 \end{pmatrix}$;
(c) $C = \begin{pmatrix} 1 & 3 & 2 \\ 5 & 4 & 4 \\ 1 & -2 & -1 \\ -3 & 1 & 3 \end{pmatrix}$.

19. Find the line $y = a + bx$ that best fits the three points $(1,1)^T$, $(3,2)^T$ and $(4,6)^T$ in the least squares sense.

20. For the points $(-1,4)$, $(0,1)$, $(1,0)$, $(2,1)$:
(a) Find the line $y = a + bx$ that is a best fit in the least squares sense;
(b) Find the quadratic $y = a + bx + cx^2$ that is a best fit in the least squares sense.

21. Let P, Q, R be the three points $(1,1)$, $(2,1)$ and $(3,1)$ in $\mathbb{R}^2$ respectively.
(a) Find the line $y = ax$ through the origin that best fits P, Q, R.
(b) Find the line $y = b + ax$ that best fits P, Q, R.
(c) Find the line $y = b + ax$ that best fits P, Q, R and the origin.

22. A farmer fertilises four fields with different amounts x of fertiliser and gets different yields y. The four yields are $\frac{1}{2}$, 1, $\frac{5}{2}$ and 3 for fertiliser amounts 0, 1, 2, and 3 respectively. Find the line of best fit though these points.

Show that this line goes through the point whose x-coordinate is the average amount of fertiliser and whose y-coordinate is the average yield. Does this always hold for any four amounts x_1, x_2, x_3, x_4 when the yields are y_1, y_2, y_3, y_4?

23. The voltage V of a discharging battery after 0, $\frac{1}{2}$, 1 and $1\frac{1}{2}$ minutes is measured as 3.5, 2, 1.5 and 1 volts respectively Find the quadratic of best fit for the voltage as a function of time.

24. Let $\mathbf{v}_1 = (0,1,2,2)^T$, $\mathbf{v}_2 = (-6,-1,-2,7)^T$, $\mathbf{v}_3 = (2,0,-9,6)^T$ be three vectors in $\mathbb{R}^4$.

Let $\mathcal{W} = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.
(a) Using the Gram-Schmidt process give an orthonormal basis for $\mathcal{W}$.
(b) Hence, or otherwise, give a QR factorisation of the matrix $A = \begin{pmatrix} 0 & -6 & 2 \\ 1 & -1 & 0 \\ 2 & -2 & -9 \\ 2 & 7 & 6 \end{pmatrix}$.
(c) Hence, or otherwise, show that the plane $ax + by + cz - w = 0$ that best fits (in the least squares sense) the four points $(x_i, y_i, z_i, w_i) = (0,-6,2,-2), (1,-1,0,2), (2,-2,-9,-3), (2,7,6,5)$ is given by $a = 5/9$, $b = 1/3$, $c = 1/3$.

25. Find the matrix of a reflection that exchanges the the vectors $\mathbf{v}$ and $\mathbf{w}$ when:
(a) $\mathbf{v} = (12,3,-5)^T$ and $\mathbf{w} = (13,0,-3)^T$;
(b) $\mathbf{v} = (-1,0,0,7)^T$ and $\mathbf{w} = (-3,2,-1,6)^T$;
(c) $\mathbf{v} = (3,-1,2,4)^T$ and $\mathbf{w} = (5,0,-1,2)^T$.
Find also the (hyper)planes fixed by the transformations.

26. Why is there no reflection that exchanges the vectors $\mathbf{v} = \begin{pmatrix} 0 \\ -9 \\ 0 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} 8 \\ 0 \\ -4 \end{pmatrix}$?

27. Let $A = \begin{pmatrix} 2 & 2 \\ 3 & 1 \\ 6 & 0 \end{pmatrix}$. Find two different QR factorisations of A, one using the Gram-Schmidt process and one by Householder transformations.

28. Use the Householder method to find a QR factorisation for $\begin{pmatrix} 1 & 1 \\ 2 & 4 \\ -2 & 3 \end{pmatrix}$.

Answers: **1.** $\frac{5x_1+12x_2}{169}(5,12)^T$. **2.** $\frac{1}{9}(5x_1 + 4x_2 - 2x_3, 4x_1 + 5x_2 + 2x_3, -2x_1 + 2x_2 + 8x_3)^T$. **3.** $(x_1,x_2,x_3) - \frac{1}{9}(5x_1 + 4x_2 - 2x_3, 4x_1 + 5x_2 + 2x_3, -2x_1 + 2x_2 + 8x_3)^T$. **4.** $\{\frac{1}{9}(-1,8,4)^T, \frac{1}{9}(8,-1,4)^T, \frac{1}{9}(4,4,-7)^T\}$. **5.** $Q = \frac{1}{9}\begin{pmatrix} -1 & 8 & 4 \\ 8 & -1 & 4 \\ 4 & 4 & -7 \end{pmatrix}$, $R = \begin{pmatrix} 9 & 9 & 9 \\ 0 & 9 & 9 \\ 0 & 0 & 9 \end{pmatrix}$. **7.** (a) no, (b) no, (c) yes, (d) no, (e) no **10.** (a) $\operatorname{span}\{(5/13, 12/13)^T, (-12/13, 5/13)^T\}$; (b) $\operatorname{span}\{(-2/3, 1/3, -2/3)^T, (5/3\sqrt{5}, 2/3\sqrt{5}, -4/\sqrt{5})^T\}$; (c) $\operatorname{span}\{(2/7, -3/7, 6/7)^T, (9/7\sqrt{2}, 4/7\sqrt{2}, -1/7\sqrt{2})^T\}$. **11.** $\frac{1}{\sqrt{2}}$, $t\sqrt{\frac{3}{2}}$, $\frac{3}{2}\sqrt{\frac{5}{2}}\left(t^2 - \frac{1}{3}\right)$, $\frac{5}{2}\sqrt{\frac{7}{2}}\left(t^3 - \frac{3}{5}t\right)$ **14.** (a) $\operatorname{span}\{(1,0,-1)^T\}$ (b) $(-4,1,5,2)^T$. **15.** (a) $\mathcal{W} = \operatorname{span}\{(0,1,1,0)^T, (3,-1,1,1)^T\}$ $\mathcal{W}^\perp = \operatorname{span}\{(1,1,-1,-1)^T, (0,1,-1,2)^T\}$ (c) $\mathbf{w}_1 \frac{1}{4}(9,-1,5,3)^T$ and $\mathbf{w}_2 = \frac{1}{4}(-5,-3,3,9)^T$. **16.** (a) $\frac{1}{5}\begin{pmatrix} 5 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & -2 & 4 \end{pmatrix}$; $(0,1,-2)^T$. (b) $\frac{1}{6}\begin{pmatrix} 5 & -2 & -1 \\ -2 & 2 & -2 \\ -1 & -2 & 5 \end{pmatrix}$; $(0,-2,4)^T$. **18.** (a) $\frac{1}{13}\begin{pmatrix} 5 & -12 \\ 12 & 5 \end{pmatrix}$, $\begin{pmatrix} 13 & 4 \\ 0 & 6 \end{pmatrix}$ (b) $\frac{1}{3}\begin{pmatrix} -2 & 1 & -2 \\ 5/\sqrt{5} & 2/\sqrt{5} & -4/\sqrt{5} \end{pmatrix}\begin{pmatrix} 3 & 6 \\ 0 & 3\sqrt{5} \end{pmatrix}$ **19.** $2y = 3x - 2$. **21.** (a) $y = 3x/7$, (b) $y = 1$, (c) $10y = 3(x+1)$. **22.** $10y = 9x + 4$, Yes. **23.** $V = T^2 - 31T/10 + 69/20$. **24.** (b) $Q = \frac{1}{9}\begin{pmatrix} 0 & -6 & 6 \\ 3 & -2 & 2 \\ 6 & -4 & -5 \\ 6 & 5 & 4 \end{pmatrix}$, $R = \begin{pmatrix} 3 & 3 & -2 \\ 0 & 9 & 6 \\ 0 & 0 & 9 \end{pmatrix}$. **25.** (a) $\frac{1}{7}\begin{pmatrix} 6 & 3 & -2 \\ 3 & -2 & 6 \\ -2 & 6 & 3 \end{pmatrix}$, $x_1 - 3x_2 + 2x_3 = 0$. (b)

$\frac{1}{5}\begin{pmatrix} 1 & 4 & -2 & -4 \\ 4 & 1 & 2 & 2 \\ -2 & 2 & 4 & -1 \\ -2 & 2 & -1 & 4 \end{pmatrix}$, $2x_1 - 2x_2 + x_3 + x_4 = 0$. (c) $\frac{1}{9}\begin{pmatrix} 5 & -2 & 6 & 4 \\ -2 & 8 & 3 & 2 \\ 6 & 3 & 0 & -6 \\ 4 & 2 & -6 & 5 \end{pmatrix}$, $2x_1 + x_2 - 3x_3 - 2x_4 = 0$. **26.** Vectors have different norms. **27.** $\frac{1}{7}\begin{pmatrix} 2 & 6 \\ 3 & 2 \\ 6 & -3 \end{pmatrix}\begin{pmatrix} 7 & 1 \\ 0 & 2 \end{pmatrix} = \frac{1}{7}\begin{pmatrix} -2 & -6 & -3 \\ -3 & -2 & 6 \\ -6 & 3 & -2 \end{pmatrix}\begin{pmatrix} -7 & -1 \\ 0 & -2 \\ 0 & 0 \end{pmatrix}$.

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PROBLEM SHEET 5

Determinants

1. (a) Use row reduction to find the determinants of the following 3×3 matrices:

$$A=\begin{pmatrix}-1&1&2\\2&4&-1\\0&-1&1\end{pmatrix},\quad B=\begin{pmatrix}28&12&0\\21&10&1\\7&2&-2\end{pmatrix},$$
$$C=\begin{pmatrix}4&0&-4\\0&4&4\\-2&-3&-1\end{pmatrix},\quad D=\begin{pmatrix}0&1&0\\0&0&1\\-2&-3&-2\end{pmatrix}.$$

(b) Use row reduction to find the determinants of the following 4×4 matrices:

$$E=\begin{pmatrix}-1&-2&-4&2\\1&1&4&-1\\-2&0&0&2\\-1&0&-1&1\end{pmatrix};\ F=\begin{pmatrix}3&1&1&-2\\-1&-6&0&0\\2&2&1&-2\\-4&-2&-1&3\end{pmatrix};$$
$$G=\begin{pmatrix}0&3&0&0\\-1&1&0&0\\0&0&5&-4\\0&0&5&-3\end{pmatrix}.$$

2. Show that $\begin{vmatrix}a&b&c&d\\-a&b&c&d\\-a&-b&c&d\\-a&-b&-c&d\end{vmatrix}=8abcd.$

3. Find all roots of the polynomial
$$p(x)=\det\begin{pmatrix}x^3&8&1\\x&2&1\\1&1&1\end{pmatrix}.$$

4. Show that $\det\begin{pmatrix}x^2&4&1\\x&2&1\\2&x&2x\end{pmatrix}=(x-1)(x-2)(3x+2).$

5. Show that the equation of the plane through three given non-collinear points (a_1,a_2,a_3), (b_1,b_2,b_3), (c_1,c_2,c_3) in $\mathbb{R}^3$ is $\det\begin{pmatrix}1&x_1&x_2&x_3\\1&a_1&a_2&a_3\\1&b_1&b_2&b_3\\1&c_1&c_2&c_3\end{pmatrix}=0.$

6. (a) Show that if A and B are invertible then $\det\left((A^T)^7B^{-15}A^{29}(B^T)^{11}A^{-36}B^4\right)=1$.

(b) Give an example to show that det is not linear (e.g. find an A and B such that $\det(A+B)\neq \det A+\det B$).

7. Let A be a 5×5 matrix with determinant 6. Let B be the matrix that results from multiplying the matrix A by -1. Let C be the matrix obtained from B by adding twice the first row of B to the third row of B. Finally let D be the matrix obtained by swapping the first and fourth columns of C and dividing the second row by 2. What are the determinants of B, C and D?

8. Suppose A is an $n\times n$ real matrix with QR-decomposition $A=QR$ where the diagonal elements of R are $\alpha_1,\alpha_2,\ldots,\alpha_n$. Show that
$$\det A=\pm\alpha_1\alpha_2\ldots\alpha_n.$$

9. For what values of x is the determinant
$$\begin{vmatrix}x&a&b&c\\a&x&b&c\\a&b&x&c\\a&b&c&x\end{vmatrix}=0?$$

10. (**H**) Suppose $A=\begin{pmatrix}M_1&N\\0&M_2\end{pmatrix}$ where M_1 and M_2 are square matrices. Show that
$$\det A=(\det M_1)(\det M_2).$$

11. (a) Show that
$$\det\begin{pmatrix}1&a&a^2\\1&b&b^2\\1&c&c^2\end{pmatrix}=(a-b)(b-c)(c-a).$$

(b) (**H**) Show that if $a_i\in\mathbb{R},\ 1\le i\le n$ then
$$\det\begin{pmatrix}1&a_1&a_1^2&\cdots&a_1^{n-1}\\1&a_2&a_2^2&\cdots&a_2^{n-1}\\\vdots&\vdots&\vdots&&\vdots\\1&a_n&a_n^2&\cdots&a_n^{n-1}\end{pmatrix}=\prod_{1\le i<j\le n}(a_j-a_i).$$

12. Let $B=\begin{pmatrix}-2&1\\3&4\\-1&5\end{pmatrix}$ and $C=\begin{pmatrix}-3&2\\1&-4\\6&2\end{pmatrix}$. Calculate the determinant of the 2×2 matrix B^TC and explain without calculation why the determinant of the 3×3 matrix BC^T is zero.

13. Let
$$A=\begin{pmatrix}0&7/2&-1\\2&1&-4\\1/2&-3&0\end{pmatrix},\quad B=\begin{pmatrix}3&-3&-2\\3&-4&-2\\-4&3&3\end{pmatrix}$$
$$C=\begin{pmatrix}1&3&-2\\0&1&-2\\0&0&1\end{pmatrix},\quad D=\begin{pmatrix}0&2&0\\1&2&3\\-1&4&-2\end{pmatrix}.$$
Calculate the determinants of A, B, C, D, AD and $B+C$.

14. Calculate the determinant of the matrix $\lambda I-D$ where D is the matrix D in the above question.

Answers: **1.** (a) -9,-28,0,-2 (b) 2,-7,15 **3.** 1,2,-3 **7.** -6,-6,3 **9.** a, b, c, $-(a+b+c)$. **12.** 546. **13.** $-\frac{1}{2}$, -1, 1, -2, 1, 12. **14.** $\lambda^3-18\lambda+2$.

School of Mathematics and Statistics, University of NSW

MATH2501 Linear Algebra

PROBLEM SHEET 6

1. Eigenvalues, Eigenvectors, Diagonalisation

1. Find the characteristic polynomials of the following matrices:

$$A = \begin{pmatrix} -1 & 1 & 2 \\ 2 & 4 & -1 \\ 0 & -1 & 1 \end{pmatrix}; \quad B = \begin{pmatrix} 28 & 12 & 0 \\ 21 & 10 & 1 \\ 7 & 2 & -2 \end{pmatrix};$$

$$C = \begin{pmatrix} 4 & 0 & -4 \\ 0 & 4 & 4 \\ -2 & -3 & -1 \end{pmatrix}; \quad D = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -3 & -2 \end{pmatrix}.$$

2. By considering the rank of $A - \lambda I$ for the following matrix/number pairs, show that the given number is an eigenvalue of the matrix.

(a) $\begin{pmatrix} 0 & -1 \\ -2 & 1 \end{pmatrix}$, -1;

(b) $\begin{pmatrix} 1 & 4 & 2 \\ 2 & 1 & -2 \\ -3 & 4 & 6 \end{pmatrix}$, 4;

(c) $\begin{pmatrix} -8 & 25 & 25 \\ -7 & 21 & 20 \\ 3 & -9 & -8 \end{pmatrix}$, 1;

(d) $\begin{pmatrix} 1 & 1-i \\ 2i & 2-i \end{pmatrix}$, $-i$.

Find an eigenvector for each of the given eigenvalues.

3. Find all eigenvalues and eigenvectors of the following matrices over $\mathbb{R}$ or $\mathbb{C}$:

(a) $\begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}$;

(b) $\begin{pmatrix} 2 & -2 \\ -2 & 5 \end{pmatrix}$;

(c) $\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$;

(d) $\begin{pmatrix} -2 & 4 \\ -5 & 7 \end{pmatrix}$;

(e) $\begin{pmatrix} 5i & -4i \\ 6i & -5i \end{pmatrix}$;

(f) $\begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$;

(g) $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{pmatrix}$;

(h) $\begin{pmatrix} 3 & 1 & 2 \\ 8 & 7 & 12 \\ -2 & -2 & -2 \end{pmatrix}$.

The results of this question will be used later (in Problem Sheets 11 & 12).

4. Diagonalise those matrices in the previous question that can be diagonalised. Which of those matrices are similar?

For each diagonalisable matrix M, find a formula for M^k, k any integer.

5. Find the characteristic polynomials of the matrices:

$$C_3 = \begin{pmatrix} 0 & 0 & -a_0 \\ 1 & 0 & -a_1 \\ 0 & 1 & -a_2 \end{pmatrix}, \quad C_4 = \begin{pmatrix} 0 & 0 & 0 & -a_0 \\ 1 & 0 & 0 & -a_1 \\ 0 & 1 & 0 & -a_2 \\ 0 & 0 & 1 & -a_3 \end{pmatrix}.$$

6. Show that if

$$M = \begin{pmatrix} 2 & -5 & 5 \\ -1 & -12 & 13 \\ -1 & -19 & 20 \end{pmatrix} \quad \text{and} \quad \mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix},$$

then $M\mathbf{x} = 7\mathbf{x}$. Let

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}.$$

Find $P^{-1}MP$.

7. Without calculating any characteristic polynomial show that 7 is an eigenvalue of the matrix $\begin{pmatrix} 2 & 5 \\ 6 & 1 \end{pmatrix}$ and calculate the other eigenvalue.

8. Verify that 2 and -3 are eigenvalues of the matrix

$$A = \begin{pmatrix} 2 & -5 & -5 \\ -4 & 8 & 4 \\ 4 & -11 & -7 \end{pmatrix}.$$

Find the third eigenvalue and eigenvectors of A for each eigenvalue.

9. Find a formula for A^n where $A = \begin{pmatrix} 2 & 5 \\ 6 & 1 \end{pmatrix}$ and n is an integer. Is your formula valid for n negative?

10. Let f denote a linear map from $\mathcal{V}$ to $\mathcal{V}$. Here $\mathcal{V}$ is a 3-dimensional vector space over $\mathbb{R}$ with basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, such that

$$f(\mathbf{v}_1) = 2\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3,$$

$$f(\mathbf{v}_2) = 2\mathbf{v}_2, \quad f(\mathbf{v}_3) = \mathbf{v}_2 + \mathbf{v}_3.$$

Is there a basis $\mathcal{B}$ for $\mathcal{V}$ such that $[f]_{\mathcal{B}}$ is a diagonal matrix? If such a basis exists, find one.

11. Let f be a linear map from $\mathbb{R}^3$ to $\mathbb{R}^3$ with matrix

$$A = \begin{pmatrix} 2 & 0 & -1 \\ 0 & -3 & 2 \\ 0 & -5 & 3 \end{pmatrix}$$

with respect to the standard basis for $\mathbb{R}^3$. Find the eigenvalues and eigenvectors of f. Do the same for the linear map from $\mathbb{C}^3$ to $\mathbb{C}^3$ that has matrix A relative to the standard basis for $\mathbb{C}^3$.

12. (a) Show that if A and B are matrices so that AB and BA are both defined then AB and BA have the same trace. Set $A = S^{-1}M$ and $B = S$ in this and show that $S^{-1}MS$ and M have the same trace.

(b) Show that

$$M = \begin{pmatrix} 12 & 13 & 14 \\ 17 & 18 & 19 \\ 23 & 24 & 70 \end{pmatrix} \quad \text{and} \quad N = \begin{pmatrix} 30 & 41 & 42 \\ 21 & 35 & -23 \\ 38 & 11 & 36 \end{pmatrix}$$

are not similar.

13. The Fibonacci numbers $\{f_n\}_{n\geq 0}$ are defined by $f_{n+2} = f_{n+1} + f_n$ for $n \geq 2$ with $f_0 = 0, f_1 = 1$.

(a) If

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix},$$

show that

$$A^n = \begin{pmatrix} f_{n-1} & f_n \\ f_n & f_{n+1} \end{pmatrix};\ n \geq 1.$$

(b) Show that $f_{n+1}f_{n-1} = f_n^2 + (-1)^n$.

(c) By diagonalising the A of (a) find a formula for f_n.

14. On the island of Kerguelen three species A, B, C of feral pigs are in mutual conflict, which is broken off each spring for breeding. The numbers a', b', c' of each species on 1 December of any year is determined in terms of the numbers a, b, c of the species on 1 December of the previous year by the formula: $a' = 7a/5 - b/10 - c/8$, $b' = -a/5 + 13b/10 - c/8$, $c' = -a/5 - b/5 + 5c/4$.

Find the population ratio corresponding to a stable population.

Answers: **1.** $\lambda^3 - 4\lambda^2 - 4\lambda + 9$; $\lambda^3 - 36\lambda^2 - 50\lambda + 28$; $\lambda^3 - 7\lambda^2 + 12\lambda$; $\lambda^3 + 2\lambda^2 + 3\lambda + 2$. **2.** (a) $\mathcal{E}_{-1} = \text{span}\{(1,1)^T\}$ (b) $\mathcal{E}_4 = \text{span}\{(2,2,-1)^T\}$ (c) $\mathcal{E}_1 = \text{span}\{(0,1,-1)^T\}$ (d) $\mathcal{E}_{-i} = \text{span}\{(i,1)^T\}$ **3.** (a) $\mathcal{E}_2 = \text{span}\{(2,1)^T\}$; $\mathcal{E}_3 = \text{span}\{(1,1)^T\}$. (b) $\mathcal{E}_1 = \text{span}\{(2,1)^T\}$; $\mathcal{E}_6 = \text{span}\{(-1,2)^T\}$. (c) $\mathcal{E}_1 = \text{span}\{(1,-1)^T\}$ (d) $\mathcal{E}_2 = \text{span}\{(1,1)^T\}$; $\mathcal{E}_3 = \text{span}\{(4,5)^T\}$. (e) $\mathcal{E}_i = \text{span}\{(1,1)^T\}$; $\mathcal{E}_{-i} = \text{span}\{(2,3)^T\}$. (f) $\mathcal{E}_0 = \text{span}\{(-1,1,1)^T\}$; $\mathcal{E}_1 = \text{span}\{(0,1,-1)^T\}$; $\mathcal{E}_3 = \text{span}\{(2,1,1)^T\}$. (g) $\mathcal{E}_2 = \text{span}\{(1,0,0)^T, (0,1,-1)^T\}$; $\mathcal{E}_4 = \text{span}\{(0,1,1)^T\}$ (h) $\mathcal{E}_2 = \text{span}\{(-2,-4,3)^T\}$; $\mathcal{E}_3 = \text{span}\{(1,4,-2)^T\}$ **4.** **a.** $P = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$ **b.** $P = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$, $D = \begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix}$. **c.** Not diagonalisable. **d.** $P = \begin{pmatrix} 1 & 4 \\ 1 & 5 \end{pmatrix}$, $D = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$. **e.** $P = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$, $D = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$. **f.** $P = \begin{pmatrix} -1 & 0 & 2 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$. **g.** $P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$. **h.** Not diagonalisable. **5.** $-\lambda^3 - \lambda^2 a_2 - \lambda a_1 - a_0$; $\lambda^4 + \lambda^3 a_3 + \lambda^2 a_2 + \lambda a_1 + a_0$. **6.** $\begin{pmatrix} 7 & -5 & 5 \\ 0 & -2 & 3 \\ 0 & -4 & 5 \end{pmatrix}$. **7.** -4. **8.** 4, $\mathcal{E}_4 = \text{span}\{(0,-1,1)^T\}$, $\mathcal{E}_{-3} = \text{span}\{(1,0,1)^T\}$, $\mathcal{E}_2 = \text{span}\{(-1,-2,2)^T\}$. **9.** $\frac{1}{11}\begin{pmatrix} 6.7^n + 5(-4)^n & 5.7^n - 5(-4)^n \\ 6.7^n - 6(-4)^n & 5.7^n + 6(-4)^n \end{pmatrix}$ **10.** No. **11.** Over $\mathbb{R}$: $\mathcal{E}_2 = \text{span}\{(1,0,0)^T\}$; Over $\mathbb{C}$, $\mathcal{E}_2$ and $\mathcal{E}_i = \text{span}\{(2+i, 3-i, 5)^T\}$; $\mathcal{E}_{-i} = \text{span}\{(2-i, 3+i, 5)^T\}$. **13.** (c) $f_n = [((1+\sqrt{5})/2)^n - ((1-\sqrt{5})/2)^n]/\sqrt{5}$. **14.** The ratio $a : b : c$ should be $2 : 3 : 4$.

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MATH2501 Linear Algebra

PROBLEM SHEET 7

Rotations and Reflections

1. Show that the following matrices are the matrices of rotations in $\mathbb{R}^2$. What are the angles of the rotations?

(a) $A = \frac{1}{5}\begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix}$

(b) $B = \frac{1}{2}\begin{pmatrix} \sqrt{3} & 1 \\ -1 & \sqrt{3} \end{pmatrix}$.

2. Show that the following matrices represent rotations in $\mathbb{R}^3$ and in each case find the axis of the rotation and the size of the angle of rotation.

(a) $A = \begin{pmatrix} -1/9 & 8/9 & 4/9 \\ 8/9 & -1/9 & 4/9 \\ 4/9 & 4/9 & -7/9 \end{pmatrix}$;

(b) $B = \begin{pmatrix} 1/2 & -1/2 & 1/\sqrt{2} \\ 1/2 & -1/2 & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \end{pmatrix}$;

(c) $C = \begin{pmatrix} \sqrt{3}/2 & 0 & -1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & \sqrt{3}/2 \end{pmatrix}$.

3. The linear maps R, S and T have matrices

$$\frac{1}{9}\begin{pmatrix} 8 & -1 & 4 \\ -1 & 8 & 4 \\ 4 & 4 & -7 \end{pmatrix}, \quad \begin{pmatrix} 1/2 & 0 & \sqrt{3}/2 \\ 0 & 1 & 0 \\ \sqrt{3}/2 & 0 & -1/2 \end{pmatrix}$$

and

$$\frac{1}{3}\begin{pmatrix} 2 & -2 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & -2 \end{pmatrix}$$

respectively with respect to the standard basis.

Show that R and S are reflections in some planes and find those planes. Show that T is a reflection in some plane composed with a rotation of that plane and find the plane and the size of the angle of rotation.

4. Determine which of the following matrices is a rotation and which is a reflection/rotation. For the rotation, determine the size of the angle and axis of rotation. For the reflection determine the normal of the plane of reflection.

$$A = \frac{1}{3}\begin{pmatrix} 2 & -2 & 1 \\ -1 & -2 & -2 \\ 2 & 1 & -2 \end{pmatrix}, \quad B = \frac{1}{9}\begin{pmatrix} 4 & -7 & -4 \\ 1 & -4 & 8 \\ 8 & 4 & 1 \end{pmatrix}.$$

5. Verify that the vector $(-3, 2, -4)^T$ is an eigenvector for the matrix

$$M = \frac{1}{15}\begin{pmatrix} -5 & -2 & 14 \\ -10 & -10 & -5 \\ 10 & -11 & 2 \end{pmatrix}.$$

Now verify efficiently the fact that M represents a rotation.

6. (**H**) Let $\mathcal{V} = \mathbb{R}^n$ with its usual inner product, and let $\mathbf{v} \in \mathcal{V}$ satisfy $\|\mathbf{v}\| = 1$. Let H denote the hyperplane with equation $\mathbf{v} \cdot \mathbf{x} = 0$. Find formulæ for:

(a) the projection onto H;

(b) the reflection through H.

Answers: **1.** (a) $\cos^{-1}(3/5)$ (b) $-\frac{\pi}{6}$ **2.** (a) axis $(2, 2, 1)^T$, angle π. (b) axis $(\sqrt{2}, 0, 1)^T$, angle $\frac{2\pi}{3}$ (c) axis $(0, 1, 0)^T$, angle $\frac{\pi}{6}$

3. R: $x_1 + x_2 - 4x_3 = 0$; S: $x_1 - \sqrt{3}x_3 = 0$; T: $x_1 + x_2 - 3x_3 = 0$, $\cos^{-1}\frac{5}{6}$. **4.** A rotation, angle $\cos^{-1}(-\frac{5}{6})$, axis $(3, -1, 1)^T$. B rotation with reflection, normal $(-1, -3, 2)^T$.

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PROBLEM SHEET 8

Symmetric Matrices

1. Find an orthonormal basis of $\mathbb{R}^3$ consisting of eigenvectors for the matrix
$$A = \begin{pmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{pmatrix}.$$
Note that A has 1 as an eigenvalue.

2. Find orthogonal matrices that diagonalise the following matrices:
$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} a & 1 & 1 & 1 \\ 1 & a & 1 & 1 \\ 1 & 1 & a & 1 \\ 1 & 1 & 1 & a \end{pmatrix}$$
$$B = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 8 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

3. Show that -1 is an eigenvalue of the matrix:
$$A = \begin{pmatrix} 0 & -1 & 2 \\ -1 & 0 & -2 \\ 2 & -2 & 3 \end{pmatrix}.$$
Find an orthogonal matrix Q such that $Q^{-1}AQ$ is diagonal.

4. Suppose that the matrix Q is orthogonal and $Q^{-1}AQ$ is diagonal. Show that A is symmetric.

5. For each of the following conic sections, find principal axes, identify the curve and sketch it:
(a) $x^2 + 4xy + y^2 = 1$;
(b) $3x^2 - 4xy + 3y^2 = 1$;
(c) $x^2 + 4xy + 4y^2 = 5$;
(d) $13x^2 + 4xy + 10y^2 = 2$.

6. For each of the conic sections in the above question find a rotation that puts the surface into canonical form and give the angles of the rotations.

7. For each of the following quadratic surfaces, find principal axes, identify the surface, sketch the surface and find the points on the surface closest to the origin :
(a) $2x^2 + 2y^2 + 5z^2 + 2xy + 4xz + 4yz = 1$;
(b) $x^2 + y^2 + 2z^2 - 2xz - 2yz = 1$;
(c) $4z^2 + 2xy + 6xz + 6yz = 1$;
(d) $x^2 + y^2 + z^2 - 4xy - 4xz - 4yz = 3$;
(e) $2x^2 - y^2 - z^2 - 4xy - 4xz - 8yz = 1$;
(f) $-8z^2 + 2xy + 4xz - 4yz = 1$
(g) $3x^2 + 5y^2 + 4z^2 + 4xz + 4yz = 28$.

8. (**H**) Consider the one-parameter set of surfaces defined by $cx^2 + (c-4)y^2 + cz^2 - 6xz = 1$. For which values of c are the surfaces
(a) ellipsoids;
(b) hyperboloids of one sheet;
(c) hyperboloids of two sheets.
(d) What are the surfaces for the remaining values of c?

Show that there is a basis for $\mathbb{R}^3$ in which all these surfaces are in standard form.

9. How many sheets has the hyperboloid in $\mathbb{R}^3$ with equation $x_1x_2 + x_1x_3 + x_2x_3 = 1$?

10. Write each of the following functions as a sum of multiples of squares of independent linear functions.
(a) $x^2 + 6xy + 4y^2$;
(b) $2x^2 - 4xy + 3y^2$;
(c) $x^2 + 8y^2 + 5z^2 - 4xy + 2xz$;
(d) $2x^2 + 7y^2 + 2z^2 - 8xy + 4xz - 10yz$;
(e) $2x^2 + 4y^2 + 5z^2 - 4xy + 4xz$;
(f) $x^2+y^2+9z^2+12w^2+4xz+2xw+4yz-2yw-6zw$;
(g) $x^2 + 4y^2 + 56z^2 + 2xy + 4xz + 28yz$.

In each case find the number of negative eigenvalues that the corresponding symmetric matrix has.

Answers: **1.** $(2/3, 2/3, 1/3)^T$, $(-1/\sqrt{2}, 1/\sqrt{2}, 0)^T$, $(1/3\sqrt{2}, 1/3\sqrt{2}, -4/3\sqrt{2})^T$ **2.** A: $1/\sqrt{2}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$; B: $\begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$; C: $\begin{pmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 0 & 0 & 1/\sqrt{2} \end{pmatrix}$; D: $\begin{pmatrix} -1/\sqrt{2} & 1/2 & 1/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \\ 1/\sqrt{2} & 1/2 & 1/2 \end{pmatrix}$. **3.** $\frac{1}{\sqrt{6}}\begin{pmatrix} \sqrt{3} & \sqrt{2} & 1 \\ \sqrt{3} & -\sqrt{2} & -1 \\ 0 & \sqrt{2} & 2 \end{pmatrix}$. **5.** For each of the following conic sections, find principal axes, identify the curve and sketch it: (a) hyperbola; axes $(1/\sqrt{2}, 1/\sqrt{2})^T$ (eigenvalue 3) and $(1/\sqrt{2}, -1/\sqrt{2})$ (eigenvalue 1). (b) ellipse ; axes $(1/\sqrt{2}, 1/\sqrt{2})^T$ (eigenvalue 1) and $(-1/\sqrt{2}, 1/\sqrt{2})$ (eigenvalue 5). (c) two lines ; axes $(1/\sqrt{5}, 2/\sqrt{5})^T$ (eigenvalue 5) and $(2/\sqrt{2}, -1/\sqrt{2})$ (eigenvalue 0). (d) ellipse; axes $(-1/\sqrt{5}, 2/\sqrt{5})^T$ (eigenvalue 9) and $(2/\sqrt{5}, 1/\sqrt{5})^T$ (eigenvalue 14). **6.** The rotation matrices are formed by writing down the axes given in the original solutions as columns in order of increasing eigenvalues. **7.** (a) $E_7 = \text{span}\{(1/\sqrt{6}, 1/\sqrt{6}, 2/\sqrt{6})^T\}$; $E_1 = \text{span}\{(1/\sqrt{2}, -1/\sqrt{2}, 0)^T, (1/\sqrt{3}, 1/\sqrt{3}, -1/\sqrt{3})^T\}$; ellipsoid, closest points $\pm 1/\sqrt{42}(1,1,2)^T$. (b) $E_0 = \text{span}\{(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})^T\}$; $E_3 = \text{span}\{(1/\sqrt{6}, 1/\sqrt{6}, -2/\sqrt{6})^T\}$; $E_1 = \text{span}\{(-1/\sqrt{2}, 1/\sqrt{2}, 0)^T\}$; elliptic cylinder, closest points $\pm(1/3\sqrt{2})(1,1,-2)^T$. (c) $E_{-2} = \text{span}\{(1/\sqrt{3}, 1/\sqrt{3}, -1/\sqrt{3})^T\}$, $E_{-1} = \text{span}\{-1/\sqrt{2}, 1/\sqrt{2}, 0)^T\}$, $E_7 = \text{span}\{(1/\sqrt{6}, 1/\sqrt{6}, 2/\sqrt{6}^T\}$, hyperboloid of 2 sheets, closest points $\pm 1/\sqrt{42}(1,1,2)^T$. (d) $E_{-3} = \text{span}\{(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})^T\}$, $E_3 =$

$\text{span}\{(1/\sqrt{2}, -1/\sqrt{2}, 0)^T, (1/\sqrt{6}, 1/\sqrt{6}, -2/\sqrt{6})^T\}$, hyperboloid of one sheet, closest points the circle $(\cos\theta)/\sqrt{2}(1,-1,0) + (\sin\theta)/\sqrt{6}(1,1,-2)^T$. (e) $E_{-6} = \text{span}\{(1/3, 2/3, 2/3)^T\}$, $E_3 = \text{span}\{(2/3, 1/3, -2/3)^T, (-2/3, 2/3, -1/3)^T\}$, hyperboloid of one sheet, closest points the circle $\frac{1}{\sqrt{3}}\cos\theta(2/3, 1/3, -2/3)^T + \frac{1}{\sqrt{3}}\sin\theta(-2/3, 2/3, -1/3)^T$. (f) $E_{-9} = \text{span}\{-1/\sqrt{18}, 1/\sqrt{18}, 4/\sqrt{18})^T\}$, $E_0 = \text{span}\{(2/3, -2/3, 1/3)^T\}$, $E_1 = \text{span}\{(1/\sqrt{2}, 1/\sqrt{2}, 0)^T\}$, hyperbolic cylinder, closest points $\pm(1/\sqrt{2}, 1/\sqrt{2}, 0)^T$ **8.** (a) $c > 4$, (b) $3 < c < 4$, (c) $-3 < c < 3$ (d) $c = 4$: elliptic cylinder; $c = 3$ hyperbolic cylinder; $c \le -3$ non-existent. **9.** 2. **10.** (a) $(x+3y)^2 - 5y^2$, One. (b) $2(x-y)^2 + y^2$, None. (c) $(x - 2y + z)^2 + 4(y + z/2)^2 + 3z^2$, None. (d) $2(x-2y+z)^2 - (y+z)^2 + z^2$, One. **e.** $2(x-y+z)^2 + 2(y+z)^2 + z^2$, None. (f) $(x + 2z + w)^2 + (y + 2z - w)^2 + (z - 3w)^2 + w^2$, None.

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MATH2501 Linear Algebra

PROBLEM SHEET 9

Powers, Cayley Hamilton Theorem

1. For the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -12 & 0 & -4 \\ 6 & -3 & 2 \end{pmatrix},$$

calculate the characteristic polynomial, $p_A(t)$. Check that $p_A(A)$ is zero.

2. Show that A and A^T have the same characteristic polynomial for any square matrix A.

3. For the matrices

$$F = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}; \quad G = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{pmatrix}$$

$$\text{and } H = \begin{pmatrix} 3 & 1 & 2 \\ 8 & 7 & 12 \\ -2 & -2 & -2 \end{pmatrix}$$

use the Cayley-Hamilton Theorem to express the 5th power and (where possible) the inverse of each matrix in terms of constants, the matrix and its square. (You have calculated the characteristic polynomials of these matrices in Problem Sheet 6.)

4. The 3×3 matrix M is known to have characteristic polynomial $\lambda^3 - 3\lambda + 2$.

(a) Find formulae for M^4, M^5 and M^{-1} as linear combinations of I, M and M^2.

(b) The column 3-vectors $\mathbf{a}(n) = \big(a(n), b(n), c(n)\big)^T$ are known to satisfy $\mathbf{a}(n+1) = M\mathbf{a}(n)$. You are given $b(1) = 1$, $b(2) = 2$, $b(3) = 3$. Find $b(5)$.

5. Write down matrices with the following characteristic polynomials:

(a) $\lambda^3 - 5\lambda^2 + 2\lambda + 9$;

(b) $\lambda^6 + \lambda^4 - 7\lambda + 1$.

6. The sequences $\{a_n\}$ and $\{b_n\}$ satisfy the recurrences $a_{n+1} = 3a_n + 4b_n$ and $b_{n+1} = 5a_n + 6b_n$. Given that $a_0 = 7$ and $a_1 = 8$ calculate a_3 without calculating b_0, b_1, b_2 or b_3.

7. Recall the following problem from Problem Sheet 6: "On the island of Kerguelen three species A, B, C of feral pigs are in mutual conflict, which is broken off each spring for breeding. The numbers a', b', c' of each species on 1 December of any year is determined in terms of the numbers a, b, c of the species on 1 December of the previous year by the formula:

$$\begin{aligned} a' &= \quad 7a/5 - b/10 - c/8 \\ b' &= -a/5 + 13b/10 - c/8 \\ c' &= \quad -a/5 - b/5 + 5c/4 \end{aligned}$$

Find the population ratio corresponding to a stable population."

Show that if a_n denotes the number of pigs of species A on 1 December in year n, then, for all n,

$$a_{n+3} = \frac{79}{20}a_{n+2} - \frac{41}{8}a_{n+1} + \frac{87}{40}a_n.$$

Answers: **1.** $\lambda^3 - 2\lambda^2$. **3.** $F^5 = 40F^2 - 39F$, F^{-1} doesn't exist. $G^5 = 208G^2 - 752G + 704I$. $G^{-1} = \frac{1}{16}(G^2 - 8G + 20I)$ $H^5 = 194H^2 - 759H + 774I$, $H^{-1} = \frac{1}{18}(H^2 - 8H + 21I)$. **4.** (a) $M^4 = 3M^2 - 2M$, $M^5 = 9M - 6I - 2M^2$ and $M^{-1} = \frac{1}{2}(3I - M^2)$ (b) 5. **5.** Among others (a) $\begin{pmatrix} 0 & 0 & -9 \\ 1 & 0 & -2 \\ 0 & 1 & 5 \end{pmatrix}$ (b) $\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 7 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$ **6.** 790.

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PROBLEM SHEET 10

Jordan Forms

1. A 12×12 matrix A has sole eigenvalue 3. It is given that $\ker(A-3I)$, $\ker(A-3I)^2$, $\ker(A-3I)^3$ and $\ker(A-3I)^4$ have dimensions 4, 7, 9 and 10 respectively. What are the possible Jordan forms of A?

2. A 13×13 matrix B has sole eigenvalue 1. It is given that $\ker(B-I)$, $\ker(B-I)^2$ and $\ker(B-I)^3$ have dimensions 4, 7 and 10 respectively. What are the possible Jordan forms of B?

3. A 20×20 matrix C has characteristic polynomial $(\lambda^2-4)^{10}$. It is given that $\ker(C-2I)$, $\ker(C-2I)^2$, $\ker(C-2I)^3$ and $\ker(C-2I)^4$ have dimensions 3,6,8,10 respectively. It is given that $\ker(C+2I)$, $\ker(C+2I)^2$, $\ker(C+2I)^3$ and $\ker(C+2I)^4$ have dimensions 3,5,7,8 respectively. What can be said about the Jordan form of C?

4. Find the Jordan forms, without necessarily calculating bases, of the matrices:

$$A = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & -3 \\ 0 & 1 & -3 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

5. Suppose that the matrix D has Jordan form

$$J_3(2) \oplus J_4(2) \oplus J_5(2) \oplus J_3(-3) \oplus J_3(-3).$$

What is the dimension of the kernel (= null space) of $(D-\lambda I)^k$ for each eigenvalue λ and all values of k.

6. Suppose that the matrix E has Jordan form

$$J_2(-2) \oplus J_4(-2) \oplus J_4(-2) \oplus J_1(1) \oplus J_5(1).$$

What is the dimension of the kernel (= null space) of $(E-\lambda I)^k$ for each eigenvalue λ and all values of k.

7. The matrix

$$A = \begin{pmatrix} -3 & 1 & 0 \\ -1 & -1 & 0 \\ 5 & 1 & 1 \end{pmatrix}$$

has two distinct eigenvalues, one of which is -2. Find the Jordan form of A and a basis with respect to which the matrix takes that form.

8. Find the Jordan forms, and bases with respect to which they take those forms, of the following matrices (over $\mathbb{R}$):

$$\begin{pmatrix} 3 & -1 & 1 \\ 2 & 0 & 1 \\ 1 & -1 & 2 \end{pmatrix}, \quad \begin{pmatrix} 2 & 2 & -1 \\ -1 & -1 & 1 \\ -1 & -2 & 2 \end{pmatrix},$$

$$\begin{pmatrix} 2 & -2 & 3 \\ 10 & -4 & 5 \\ 5 & -4 & 6 \end{pmatrix}, \quad \begin{pmatrix} 2 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & -1 & 1 \end{pmatrix}.$$

9. Are the matrices

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 1 \end{pmatrix} \quad \text{and} \quad \frac{1}{2}\begin{pmatrix} 2 & -5 & 1 \\ 0 & -1 & 3 \\ 0 & 1 & 1 \end{pmatrix}$$

similar?

10. Which of the following matrices are similar:

$$A = \begin{pmatrix} 4 & 1 & 4 \\ -4 & 0 & -7 \\ 0 & 0 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 0 & 0 \\ 2 & 5 & 1 \\ -6 & -9 & -1 \end{pmatrix},$$

$$C = \begin{pmatrix} -1 & -2 & 3 \\ -2 & 0 & 2 \\ -4 & -3 & 6 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ -2 & 2 & 4 \end{pmatrix}.$$

11. Find the Jordan form J and a transition matrix M such that $J = M^{-1}AM$ for

$$A = \begin{pmatrix} -2 & 1 & 3 & -1 \\ 3 & 0 & -2 & 2 \\ 1 & 1 & 2 & 1 \\ 1 & -1 & -3 & 0 \end{pmatrix}.$$

12. Show that

$$A = \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$$

has only one Jordan block (of size 3) if and only if $ab \neq 0$.

13. Let A_1, A_2 be two 2×2 matrices, $A = A_1 \oplus A_2$ and

$$P = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix},$$

where I is the 2×2 identity matrix. Find $P^{-1}AP$ and hence show that $A_1 \oplus A_2$ is similar to $A_2 \oplus A_1$.

14. Show that $J_2(0)^2$ is the zero matrix. Find $J_3(0)^2$, and show that $J_3(0)^3$ is the zero matrix.

Generalise these results to $J_n(0)$.

15. One of the eigenvalues of

$$E = \begin{pmatrix} 5 & -2 & 1 \\ 4 & -1 & 1 \\ -4 & 2 & 0 \end{pmatrix}$$

is 1. Is E diagonalisable or not?

16. Given that the matrix C has Jordan form $J_1(4) \oplus J_2(4) \oplus J_4(4) \oplus J_4(4)$, calculate $\dim\ker(A-4I)^n$ for each integer n.

17. Given that the matrix
$$D = \begin{pmatrix} 3 & 1 & 0 & 0 \\ 0 & 3 & -1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & -1 & 4 \end{pmatrix}$$
has an eigenvalue $\lambda = 3$, find all the eigenvalues of D. Find a basis for $\mathbb{R}^4$ relative to which the matrix of the linear transformation represented by D has a matrix in Jordan form.

Answers: **1.** $J_1(3) \oplus J_2(3) \oplus J_3(3) \oplus J_6(3)$. **2.** $J_1(1) \oplus J_3(1) \oplus J_3(1) \oplus J_6(1)$ or $J_1(1) \oplus J_3(1) \oplus J_4(1) \oplus J_5(1)$ or $J_1(1) \oplus J_4(1) \oplus J_4(1) \oplus J_4(1)$. **3.** It is $J_2(2) \oplus J_4(2) \oplus J_4(2) \oplus J_1(-2) \oplus J_3(-2) \oplus J_6(-2)$. **4.** $J = J_3(-1)$. $J_3(0)$. **5.** For $k = 1, 2, 3, 4, \geq 5$ respectively $\dim\ker(D - 2I)^k = 3, 6, 9, 11, 12$, and for $k = 1, 2, \geq 3$ respectively $\dim\ker(D + 3I)^k = 2, 4, 6$. **6.** For $k = 1, 2, 3, \geq 4$ respectively $\dim\ker(E + 2I)^k = 3, 6, 8, 10$, and for $k = 1, 2, 3, 4, \geq 5$ respectively $\dim\ker(E - I)^k = 2, 3, 4, 5, 6$. **7.** Jordan form is $J_2(-2) \oplus J_1(1)$, possible basis $\{(-2, -2, 4)^T, (1, -1, 0)^T, (0, 0, 1)^T\}$. **8.** $J_1(1) \oplus J_2(2)$. $J_1(1) \oplus J_2(1)$. $J_1(2) \oplus J_2(1)$. $J_2(2) \oplus J_1(2)$. **9.** Yes: they have the same Jordan form $J_1(-1) \oplus J_2(1)$. **10.** B and D. **11.** $J = J_2(-1) \oplus J_2(1)$.

SCHOOL OF MATHEMATICS AND STATISTICS, UNIVERSITY OF NSW

MATH2501 LINEAR ALGEBRA

PROBLEM SHEET 11

MATRIX EXPONENTIAL

1. Let
$$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}.$$
Find
$$AB,\ BA,\ A^2 + 2AB + B^2,\ (A+B)^2,$$
and
$$e^A \cdot e^B,\ e^{A+B},\ e^B \cdot e^A.$$
Which of these expressions might you have thought (**a priori**) to have been equal? Explain why they are not equal.

2. Let
$$A = \begin{pmatrix} 7 & 8 \\ -6 & -7 \end{pmatrix} \text{ and } B = \begin{pmatrix} 3 & 0 & 4 \\ 2 & 1 & 4 \\ -2 & 0 & -3 \end{pmatrix}.$$
Show that A and B are both diagonalisable.

Hence find expressions for A^n and B^n for $(n \geq 0)$ and calculate $\exp(tA)$ and $\exp(tB)$.

3. Let A be a square matrix which satisfies $A^3 = A^2$. Show that $\exp A = I + A + (e-2)A^2$. Hence find
$$\exp \begin{pmatrix} 0 & 2 & -1 \\ 1 & 2 & -2 \\ 0 & 2 & -1 \end{pmatrix}.$$

4. Suppose A is a square matrix and t is a scalar. Using a result about exponentials of matrices from lectures show that $\exp(tA)$ is invertible and $\exp(tA)^{-1} = \exp(-tA)$.

5. (a) (**H**) Show that if A is skew-symmetric ($A^T = -A$), then $\exp(A)$ is orthogonal.

(b) (**H**) Give an example of an orthogonal matrix Q for which there is no skew-symmetric A with $Q = \exp(A)$.

6. Calculate $\exp(tA)$ for each of the following diagonalisable matrices:

(a) $A = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}$;

(b) $A = \begin{pmatrix} 2 & -2 \\ -2 & 5 \end{pmatrix}$;

(c) $A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$;

(d) $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{pmatrix}$.

7. For each of the following matrices A, find $\exp(tA)$ by any method, and check that your answer becomes the identity matrix when you substitute $t = 0$:

(a) $\begin{pmatrix} 1 & 1 \\ -9 & -5 \end{pmatrix}$

(b) $\begin{pmatrix} 3 & -1 & 1 \\ 2 & 0 & 1 \\ 1 & -1 & 2 \end{pmatrix}$

(c) $\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

(d) $\begin{pmatrix} -1 & 1 & -1 \\ 0 & 1 & -4 \\ 0 & 1 & -3 \end{pmatrix}$.

8. Let A be the matrix
$$\begin{pmatrix} 2 & 4 & 3 \\ 4 & 3 & 4 \\ -3 & -4 & -4 \end{pmatrix}.$$
Find a basis for $\mathbb{R}^3$ consisting of generalised eigenvectors for A, and use it to find e^{tA}.

9. Show that if A is an $n \times n$ matrix, then $\exp tA \to 0$ as $t \to \infty$ whenever all the eigenvalues λ of A satisfy $\mathrm{Re}(\lambda) < 0$.

10. Calculate $\exp(tB)$ for the matrix $B = \begin{pmatrix} 5 & 1 \\ 3 & 3 \end{pmatrix}$.

11. Calculate $\exp(tA)$ for the matrix $A = \begin{pmatrix} 5 & 1 \\ -1 & 3 \end{pmatrix}$.

12. Let
$$M = \begin{pmatrix} 3 & -1 & 1 \\ 2 & 0 & 1 \\ 1 & -1 & 2 \end{pmatrix}.$$
Given that $\lambda = 2$ is a repeated eigenvalue of M, find the other eigenvalue of M. Find an invertible matrix C such that $C^{-1}MC$ is in Jordan form. Compute e^{tM}.

Answers: **1.** $AB = B$; $BA = A$; $A^2 + 2AB + B^2 = \begin{pmatrix} 4 & -2 \\ 0 & 0 \end{pmatrix}$; $(A+B)^2 = \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix}$; $\exp A \cdot \exp B) = \begin{pmatrix} e^2 & -(e-1)^2 \\ 0 & 1 \end{pmatrix}$; $\exp(A+B) = \begin{pmatrix} e^2 & 0 \\ 0 & 1 \end{pmatrix}$; $\exp B \cdot \exp A = \begin{pmatrix} e^2 & (e-1)^2 \\ 0 & 1 \end{pmatrix}$. **2.** A^n is I if n is even, A if n is odd. **3.** $\begin{pmatrix} 2e-3 & 2e-2 & 5-3e \\ 2e-3 & 2e-1 & 4-3e \\ 2e-4 & 2e-2 & 6-3e \end{pmatrix}$. **6.** (a) $\begin{pmatrix} 2e^{2t} - e^{3t} & 2e^{3t} - 2e^{2t} \\ -e^{3t} + e^{2t} & -e^{2t} + 2e^{3t} \end{pmatrix}$; (b) $\frac{1}{5}\begin{pmatrix} 4e^t + e^{6t} & -2e^{6t} + 2e^t \\ -2e^{6t} + 2e^t & e^t + 4e^{6t} \end{pmatrix}$;

(c) $\begin{pmatrix} 1/3 + 2/3\, e^{3\,t} & 1/3\, e^{3\,t} - 1/3 & 1/3\, e^{3\,t} - 1/3 \\ 1/3\, e^{3\,t} - 1/3 & 1/6\, e^{3\,t} + 1/2\, e^t + 1/3 & 1/6\, e^{3\,t} - 1/2\, e^t + 1/3 \\ 1/3\, e^{3\,t} - 1/3 & 1/6\, e^{3\,t} - 1/2\, e^t + 1/3 & 1/6\, e^{3\,t} + 1/2\, e^t + 1/3 \end{pmatrix}$

(d) $\begin{pmatrix} e^{2\,t} & 0 & 0 \\ 0 & 1/2\, e^{4\,t} + 1/2\, e^{2\,t} & -1/2\, e^{2\,t} + 1/2\, e^{4\,t} \\ 0 & -1/2\, e^{2\,t} + 1/2\, e^{4\,t} & 1/2\, e^{4\,t} + 1/2\, e^{2\,t} \end{pmatrix}$ **7.** (a)

$e^{-2t}\begin{pmatrix}1+3t & t\\ -9t & 1-3t\end{pmatrix}$. (b) $\begin{pmatrix}e^{2t}(t+1) & -te^{2t} & te^{2t}\\ (t+1)e^{2t}-e^{t} & e^{t}-te^{2t} & te^{2t}\\ e^{2t}-e^{t} & e^{t}-e^{2t} & e^{2t}\end{pmatrix}$.
(c) $e^{-t}\begin{pmatrix}1 & t & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$. (d) $e^{-t}\begin{pmatrix}1 & t+t^2/2 & -(t+t^2)\\ 0 & 2t+1 & -4t\\ 0 & t & 1-2t\end{pmatrix}$. **8.**

$$\exp tA = \begin{pmatrix} e^{3t}-te^{-t} & e^{3t}-e^{-t} & e^{3t}-(t+1)e^{-t}\\ e^{3t}-e^{-t} & e^{3t} & e^{3t}-e^{-t}\\ (t+1)e^{-t}-e^{3t} & e^{-t}-e^{3t} & (t+2)e^{-t}-e^{3t}\end{pmatrix}$$

School of Mathematics and Statistics, University of NSW

MATH2501 Linear Algebra

PROBLEM SHEET 12

Systems of linear differential equations

1. For each matrix from Problem Sheet 11, Question 6 a) and b), solve the system of differential equations given by $(y)'(t) = A(y)$ with initial conditions $\mathbf{y}(0) = (1,3)^T$.

2. For each matrix from Problem Sheet 11, Question 6 c) and d), solve the system of differential equations given by $(y)'(t) = A(y)$ with initial conditions $(y)(0) = (1,2,2)^T$.

3. Solve the system: Solve the system:

$$\begin{aligned} y_1' &= 2y_1 - y_2 + y_3, \\ y_2' &= -y_1 + 2y_2 - y_3, \\ y_3' &= y_1 - y_2 + 2y_3. \end{aligned}$$

4. Solve the system:

$$\begin{aligned} y_1' &= 3y_1 - y_2 + y_3, \\ y_2' &= 2y_1 + y_3, \\ y_3' &= 2y_1 - y_2 + 2y_3. \end{aligned}$$

with initial conditions $y_1(0) = y_3(0) = 1,\ y_2(0) = 0$.

5. (a) Find the general solution of the system

$$\mathbf{y}'(t) = A\mathbf{y}(t), \quad A = \begin{pmatrix} 2 & 6 \\ 2 & 3 \end{pmatrix}.$$

(b) Check that the non-homogeneous system

$$\mathbf{y}'(t) = A\mathbf{y} + \mathbf{b}(t), \quad \mathbf{b}(t) = \begin{pmatrix} -2t+7 \\ -2t+3 \end{pmatrix}$$

has a solution

$$\mathbf{y}(t) = \mathbf{f}(t) = \begin{pmatrix} t \\ -1 \end{pmatrix}.$$

Write down the general solution.

(c) Find the solution of

$$\mathbf{y}' = A\mathbf{y}(t) + \mathbf{c}(t),$$

with

$$\mathbf{c}(t) = \begin{pmatrix} 3e^{2t} \\ 2e^{2t} \end{pmatrix} \quad \text{and} \quad \mathbf{y}(0) = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

6. Repeat the above question with the following changes:

(a)

$$A = \begin{pmatrix} 3 & 6 \\ 1 & 2 \end{pmatrix}, \quad \mathbf{b}(t) = \begin{pmatrix} -6t^2 \\ 2t - 2t^2 \end{pmatrix},$$

$$\mathbf{f}(t) = \begin{pmatrix} 0 \\ t^2 \end{pmatrix}, \ \mathbf{c}(t) = \begin{pmatrix} -6e^{5t} \\ 3e^{5t} \end{pmatrix}, \ \mathbf{y}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

(b)

$$A = \begin{pmatrix} 11 & 9 & -14 \\ 2 & 4 & -2 \\ 12 & 12 & -15 \end{pmatrix}, \quad \mathbf{b}(t) = \begin{pmatrix} e^{2t} \\ -e^{2t} \\ 0 \end{pmatrix},$$

$$\mathbf{f}(t) = \begin{pmatrix} te^{2t} \\ -te^{2t} \\ 0 \end{pmatrix}, \ \mathbf{c}(t) = \begin{pmatrix} 6e^{t} \\ -2e^{t} \\ 3e^{t} \end{pmatrix}, \ \mathbf{y}(0) = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}.$$

7. Solve the system:

$$\begin{aligned} y_1' &= 2y_1 + y_2 + y_3 + e^{5t}, \\ y_2' &= 2y_1 + 3y_2 + 2y_3, \\ y_3' &= y_1 + y_2 + 2y_3 - e^{5t} \end{aligned}$$

with initial conditions $y_1(0) = 1/2,\ y_2(0) = 1/2,\ y_3(0) = 0$.

8. Solve the system $\mathbf{y}' = A\mathbf{y} + \mathbf{b}(t)$, where

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}, \quad \mathbf{b}(t) = e^{2t}\begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix}$$

with initial conditions $y_1(0) = 1$, $y_2(0) = 2$, $y_3(0) = 3$.

9. Let

$$A = \begin{pmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ 2 & 2 & 1 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

(a) Find $(A-3I)\mathbf{v}$ and $(A-3I)^2\mathbf{v}$.
(b) Show that the eigenvalues of A are 2 and 3.
(c) Find $\exp(tA)\mathbf{v}$.
(d) Solve the system $\mathbf{y}' = A\mathbf{y} + (e^{2t}, 0, 2e^{2t})^T$

10. Solve the system

$$\begin{aligned} y_1'(t) &= -3y_1 + y_2 + 3e^{-2t} \\ y_2'(t) &= -y_1 - y_2 + 2e^{-2t} \end{aligned}$$

where $y_1(0) = 1$, $y_2(0) = 1$.

11. (a) Show that

$$A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & -1 & 0 \\ -1 & 0 & 4 \end{pmatrix}$$

has two distinct eigenvalues and then find $\exp(tA)$.

(b) Find a basis for the vector space of solutions of $\frac{d\mathbf{y}}{dt} = A\mathbf{y}$.

12. A solution to a system of DE's is called **stable** if it approaches a limit as $t \to \infty$.

Show that all solutions of the system

$$\mathbf{y}' = \begin{pmatrix} -2 & 1 \\ -1 & -2 \end{pmatrix}\mathbf{y}$$

are stable. What feature of the matrix accounts for the stability?

13. (Exam 2006, Session 1) Let

$$A = \begin{pmatrix} 7 & 3 & 1 \\ -8 & -3 & -1 \\ -3 & -2 & 2 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix}.$$

Given that A has only one eigenvalue 2,

(a) Find the Jordan chain of A starting with the vector $\mathbf{b}$.

(b) Calculate $e^{tA}\mathbf{v}$ for each non-zero vector $\mathbf{v}$ in the chain in i) above. You may leave these as linear combinations of the non-zero vectors of the chain in i).

(c) Solve the system of differential equations

$$\mathbf{x}' = A\mathbf{x} + e^{2t}\,\mathbf{b}, \quad \mathbf{x} = \mathbf{x}(t)$$

$$\mathbf{x}(0) = \mathbf{0} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

14. (Exam 2005, Session 2) Let

$$G = \begin{pmatrix}-3 & 1 & 0\\ -4 & -1 & -1\\ 0 & -3 & -3\end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix}-1\\0\\4\end{pmatrix}, \quad \mathbf{w} = \begin{pmatrix}1\\2\\-3\end{pmatrix}.$$

(a) Show that $\mathbf{v}$ is an eigenvector of G and give its eigenvalue.

(b) Show that $\mathbf{w}$ is a generalised eigenvector of G corresponding to eigenvalue -2.

(c) Give the Jordan form J of G and a matrix P such that $J = P^{-1}GP$.

(d) Calculate $\exp(tG)\mathbf{v}$ and $\exp(tG)\mathbf{w}$.

(e) Solve the system of differential equations

$$\mathbf{y}'(t) = G\mathbf{y} + e^{-2t}\mathbf{w}$$

subject to the initial conditions $\mathbf{y}(0) = \mathbf{v}$.

Answers: **1.** $5e^{3t}\begin{pmatrix}1\\1\end{pmatrix} - 2e^{2t}\begin{pmatrix}2\\1\end{pmatrix}$; $e^{t}\begin{pmatrix}2\\1\end{pmatrix} + e^{6t}\begin{pmatrix}-1\\2\end{pmatrix}$. **2.** $\begin{pmatrix}-1\\1\\1\end{pmatrix} + e^{3t}\begin{pmatrix}2\\1\\1\end{pmatrix}$; $e^{2t}\begin{pmatrix}1\\0\\0\end{pmatrix} + 2e^{4t}\begin{pmatrix}0\\1\\1\end{pmatrix}$. **3.** $y(t) = \begin{pmatrix}a_1e^t + a_2e^t + a_3e^{4t}\\ a_1e^t - a_3e^{4t}\\ -a_2e^t + a_3e^{4t}\end{pmatrix}$ **4.** $y(t) = -1/2e^{t}\begin{pmatrix}1\\3\\1\end{pmatrix} + 3/2e^{3t}\begin{pmatrix}1\\1\\1\end{pmatrix}$ **5.** (a) $\alpha e^{6t}\begin{pmatrix}3\\2\end{pmatrix} + \beta e^{-t}\begin{pmatrix}2\\-1\end{pmatrix}$; (b) $\alpha e^{6t}\begin{pmatrix}3\\2\end{pmatrix} + \beta e^{-t}\begin{pmatrix}2\\-1\end{pmatrix} + \begin{pmatrix}t\\-1\end{pmatrix}$; (c) $e^{-t}\begin{pmatrix}-2\\1\end{pmatrix} + \frac{1}{4}e^{6t}\begin{pmatrix}15\\10\end{pmatrix} - \frac{1}{4}e^{2t}\begin{pmatrix}3\\2\end{pmatrix}$. **6.** (a)$\mathbf{y} = c_1\begin{pmatrix}2\\-1\end{pmatrix} + c_2e^{5t}\begin{pmatrix}3\\1\end{pmatrix}$; $c_1\begin{pmatrix}2\\-1\end{pmatrix} + c_2e^{5t}\begin{pmatrix}3\\1\end{pmatrix} + \begin{pmatrix}0\\t^2\end{pmatrix}$; $\mathbf{y} = \begin{pmatrix}0\\e^{5t}\end{pmatrix}$. (b) $\mathbf{y}(t) = c_1e^{-3t}\begin{pmatrix}1\\0\\1\end{pmatrix} + c_2e^{2t}\begin{pmatrix}1\\-1\\0\end{pmatrix} + c_3e^{t}\begin{pmatrix}6\\-2\\3\end{pmatrix}$; $\mathbf{y}(t) = e^{-3t}\begin{pmatrix}1\\0\\1\end{pmatrix} + e^{2t}\begin{pmatrix}1\\-1\\0\end{pmatrix} + te^{t}\begin{pmatrix}6\\-2\\3\end{pmatrix}$. **7.** $\mathbf{y} = e^{5t}(1/2, 1/2, 0)^T$ **8.** $y_1 = (t^3 + 3t^2 + 3t + 1)e^{2t}$; $y_2 = (3t^2 + 6t + 2)e^{2t}$; $y_3 = (3 + 6t)e^{2t}$. **9.** (a) $\begin{pmatrix}1\\1\\2\end{pmatrix}$, $\begin{pmatrix}0\\0\\0\end{pmatrix}$. (c) $e^{3t}\begin{pmatrix}1+t\\1+t\\1+2t\end{pmatrix}$. (d) $e^{3t}\begin{pmatrix}1+t\\1+t\\1+2t\end{pmatrix} + e^{2t}\begin{pmatrix}t\\0\\2t\end{pmatrix}$. **11.** (a) $\exp(tA) = \begin{pmatrix}(1-t)e^{3t} & 0 & te^{3t}\\ 0 & e^{-t} & 0\\ -te^{3t} & 0 & (1+t)e^{3t}\end{pmatrix}$. **12.** $\mathbf{x} = \begin{pmatrix}ae^{-2t}\sin t + be^{-2t}\cos t\\ ae^{-2t}\cos t - be^{-2t}\sin t\end{pmatrix}$. The real parts of the eigenvalues are negative.