1 problem set eight

1. take the Lagrangian

$$L = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\dot{y}^2 - x^2 - y^2 - 2xy \tag{1.1}$$

and show that it is invariant under the shifts

$$\tilde{x} = x + \epsilon, \quad \tilde{y} = y - \epsilon \tag{1.2}$$

then calculate the corresponding Nöther constant.

2. consider the simple harmonic oscillator with Lagrangian

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 \tag{1.3}$$

and consider the change

$$\tilde{t} = t \tag{1.4}$$
$$\tilde{x} = x + \epsilon \exp(-i\omega t) \tag{1.5}$$

where $k = \omega^2 m$.

- Calculate the change in the Lagrangian, show it is a total derivative, and find the corresponding conserved quantity.
- by finding the equations of motion for the system, use direct substitution to show that

$$(\dot{x} + i\omega x)\exp(-i\omega t) \tag{1.6}$$

is constant.

3. consider the Lagrangian

$$L = \frac{1}{2}\dot{x}^2 + \frac{k}{x^2} \tag{1.7}$$

- show that this system leaves the action invariant, in the sense given in the lectures, under the scaling transformation

$$\tilde{t} = \lambda t \tag{1.8}$$
$$\tilde{x}(\tilde{t}) = \sqrt{\lambda}x(t) \tag{1.9}$$

- take $\lambda = 1 + \epsilon$ to find the conserved quantity associated with this scaling symmetry.
- show that

$$-Et + \frac{1}{2}x\dot{x} \tag{1.10}$$

is constant, where $E = \frac{1}{2}\dot{x}^2 - \frac{k}{x^2}$ is the total energy, which is constant.

2 solution set seven

1. substitution shows that the Lagrangian is invariant, meaning that $G = 0$. Now take the definition of the proposed symmetry and compare this to the general form to see that $\xi = 0$, $\eta^{(x)} = 1$, $\eta^{(y)} = -1$, then using the expression for the Nöther constant gives

$$\dot{x} - \dot{y} = const \tag{2.11}$$

Here we have used a generalization of the Nöther theorem we derived in the lectures, so that we may apply it to multiple variables

$$\xi L - G + \sum_{(\alpha)} (\eta^{(\alpha)} - \xi \dot{q}^{(\alpha)}) \frac{\partial L}{\partial \dot{q}^{(\alpha)}} \quad = \quad 0 \tag{2.12}$$

where

$$\begin{aligned} \tilde{t} &= t + \epsilon \xi(t) && (2.13) \\ \tilde{q}^{(\alpha)}(\tilde{t}) &= q^{(\alpha)}(t) + \epsilon \eta^{(\alpha)}(q, t) && (2.14) \end{aligned}$$

2. with the suggested substitutions we note that to first order in ϵ

$$L(\tilde{t}, \tilde{x}) \quad = \quad L(x,t) - im\omega\epsilon \left[\dot{x}e^{-i\omega t} - i\omega x e^{-i\omega t}\right] = L(x,t) - im\omega\epsilon \frac{d}{dt}\left[x e^{-i\omega t}\right] \tag{2.15}$$

so we have

$$\xi = 0, \quad \eta = e^{-i\omega t}, \quad G = -im\omega x e^{-i\omega t} \tag{2.16}$$

and the Nöther constant is as advertised. It is important to note that in finding $L(\tilde{x})$ we simply replace x by $\tilde{x}$, one does not substitute for x using $x = \tilde{x} - \epsilon e^{-i\omega t}$. i.e.

$$\begin{aligned} L(\tilde{x}) &= \frac{1}{2} m \dot{\tilde{x}}^2 - \frac{1}{2} k \tilde{x}^2 \quad \textit{is CORRECT} && (2.17) \\ Ł(\tilde{x}) &= \frac{1}{2} m (\frac{d}{dt}[\tilde{x} - \epsilon e^{-i\omega t}])^2 - \frac{1}{2} \tilde{k(}\tilde{x} - \epsilon e^{-i\omega t})^2 \quad \textit{is INCORRECT} && (2.18) \end{aligned}$$

Checking that this quantity is a constant using the equations of motion simply means differentiate it with respect to t, (this will introduce an $\ddot{x}$ term), then use the equation of motion to see that it vanishes, i.e.

$$\begin{aligned} \frac{d}{dt}\left[(\dot{x} + i\omega x)\exp(-i\omega t)\right] &= (\ddot{x} + i\omega\dot{x})\exp(-i\omega t) - i\omega(\dot{x} + i\omega x)\exp(-i\omega t) \\ &= (\ddot{x} + \omega^2 x)\exp(-i\omega t) \\ &= 0 \end{aligned}$$

where the last line follows from the equation of motion $\ddot{x} + \omega^2 x = 0$.

3. Using the given proposed symmetry we have

$$\int d\tilde{t} L(\tilde{t}, \tilde{x}) = \int \lambda dt \left[\frac{1}{2\lambda} \left(\frac{dx}{dt} \right)^2 + \frac{k}{\lambda x^2} \right] \tag{2.19}$$

$$= \int dt \left[\frac{1}{2} \left(\frac{dx}{dt} \right)^2 + \frac{k}{x^2} \right] \tag{2.20}$$

$$= \int dt \; L(t, x) \tag{2.21}$$

Where again, we remember that to get $L(\tilde{x})$ we just replace x by $\tilde{x}$, we do not substitute for it. For $\lambda = 1 + \epsilon$ we see

$$\tilde{t} = t + \epsilon t, \quad \tilde{x} = x + \frac{1}{2} \epsilon x \tag{2.22}$$

giving $\xi = t$, $\eta = \frac{1}{2}x$, $G = 0$, and a Nöther constant of

$$t \left[\frac{1}{2} \dot{x}^2 + \frac{k}{x^2} \right] + \left[\frac{1}{2} x - t\dot{x} \right] \dot{x} = const \tag{2.23}$$

$$\Rightarrow -t \left[\frac{1}{2} \dot{x}^2 - \frac{k}{x^2} \right] + \frac{1}{2} x\dot{x} = const \tag{2.24}$$

$$\Rightarrow -Et + \frac{1}{2} x\dot{x} = const \tag{2.25}$$

that this is a constant may be checked explicitly with the equations of motion.