102. Construct the lower sum L and the upper sum U that belong to the subdivision $a = x_0 < x_1 < \cdots < x_n = b$ (as described on p. 46):

$$U = \sum_{\nu=1}^{n} M_\nu (x_\nu - x_{\nu-1}), \qquad L = \sum_{\nu=1}^{n} m_\nu (x_\nu - x_{\nu-1}).$$

By a proper choice of the subdivision we can attain that $U - L < \varepsilon$. Now we define $\Psi(x)$ as follows: $\Psi(x) = M_\nu$ on $[x_{\nu-1}, x_\nu)$, $\nu = 1, 2, \ldots, n-1$, $\Psi(x) = M_n$ on $[x_{n-1}, x_n]$. We define $\psi(x)$ similarly using m_ν. Then

$$\int_a^b \Psi(x)\, dx = U, \qquad \int_a^b \psi(x)\, dx = L.$$

The only condition imposed on the subdividing points is that the maximal length of the subintervals $[x_{\nu-1}, x_\nu]$, $\nu = 1, 2, \ldots, n$, converges to 0 as n increases. Therefore these points can be chosen equidistant, forming an arithmetic progression. The functions $\Psi(x)$ and $\psi(x)$ constructed in the described way are continuous on the right; they could be defined continuous on the left instead.

103. Define $\Psi(x)$ and $\psi(x)$ as in solution **102**. Then the total variation of $\Psi(x)$ is

$$|M_2 - M_1| + |M_3 - M_2| + \cdots + |M_n - M_{n-1}|$$

and that of $\psi(x)$ is

$$|m_2 - m_1| + |m_3 - m_2| + \cdots + |m_n - m_{n-1}|.$$

Both are not larger than the total variation of $f(x)$ because $f(x)$ assumes on $[x_{\nu-1}, x_\nu]$ values which are arbitrarily close to M_ν and m_ν.

104. Let ν be an integer, $\nu = 1, 2, \ldots, n$; in the first half of the interval $\left[\frac{\nu-1}{n}, \frac{\nu}{n}\right]$ we have $s(nx) = +1$, in the second half $s(nx) = -1$. Thus

$$\int_0^1 f(x)\, s(nx)\, dx = \int_0^{\frac{1}{2n}} \sum_{\nu=1}^{n} \left\{ f\left(\frac{\nu-1}{n} + y\right) - f\left(\frac{\nu-1}{n} + y + \frac{1}{2n}\right) \right\} dy.$$

The absolute value of the expression between the curly brackets is smaller than the oscillation of $f(x)$ on $\left[\frac{\nu-1}{n}, \frac{\nu}{n}\right]$.

105. [Riemann: Werke. Leipzig: B. G. Teubner 1876, p. 240; E. W. Hobson: The Theory of Functions of a Real Variable & The Theory

Part Two

Integration

Chapter 1

The Integral as the Limit of a Sum of Rectangles

§ 1. The Lower and the Upper Sum

Let $f(x)$ be a bounded function on the finite interval $[a, b]$. The points with abscissae $x_0, x_1, x_2, \ldots, x_{n-1}, x_n$, where

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b,$$

constitute a subdivision of this interval. Denote by m_ν and M_ν the greatest lower and the least upper bound of $f(x)$ on the ν-th subinterval $[x_{\nu-1}, x_\nu]$, $\nu = 1, 2, \ldots, n$. We call

$$L = \sum_{\nu=1}^{n} m_\nu (x_\nu - x_{\nu-1}) \quad \text{the } \textit{lower sum},$$

$$U = \sum_{\nu=1}^{n} M_\nu (x_\nu - x_{\nu-1}) \quad \text{the } \textit{upper sum}$$

belonging to the subdivision $x_0, x_1, x_2, \ldots, x_{n-1}, x_n$. Any upper sum is always larger (not smaller) than any lower sum, regardless of the subdivision considered. If there exists *only one* number which is neither smaller than any lower sum nor larger than any upper sum, then this number, denoted by the symbol

$$\int_a^b f(x)\, dx,$$

is called the *definite integral* of $f(x)$ over the interval $[a, b]$ and $f(x)$ is called (properly) integrable over $[a, b]$ in the sense of Riemann.