If $\max (x_\nu - x_{\nu-1}) \to 0$ (the subinterval of maximal length converges to 0) we obtain the relation

$$\lim_{n\to\infty} \sum_{\nu=1}^{n} f(y_\nu)\, \varphi(\eta_\nu)\, (x_\nu - x_{\nu-1}) = \int_a^b f(x)\, \varphi(x)\, dx.$$

101. Suppose that $f(x)$ is properly integrable over $[a, b]$ and $\varphi(x)$ properly integrable over $[a, b + d]$, $d > 0$. Then

$$\lim_{\delta\to+0} \int_a^b f(x)\, \varphi(x + \delta)\, dx = \int_a^b f(x)\, \varphi(x)\, dx.$$

102. Let $f(x)$ denote a properly integrable function on $[a, b]$. There exist to every positive number ε two *step-functions*, $\psi(x)$ and $\Psi(x)$, such that for the entire interval $[a, b]$

$$\psi(x) \leqq f(x) \leqq \Psi(x)$$

and

$$\int_a^b \Psi(x)\, dx - \int_a^b \psi(x)\, dx < \varepsilon.$$

It is even possible to choose $\psi(x)$ and $\Psi(x)$ so that their points of discontinuity are equidistant.

103 (continued). If $f(x)$ is of bounded variation $\psi(x)$ and $\Psi(x)$ may be chosen so that the total variation of neither exceeds the total variation of $f(x)$.

104. We define

$$4[x] - 2[2x] + 1 = s(x).$$

Then we have (n integer) the limit relation

$$\lim_{n\to\infty} \int_0^1 f(x)\, s(nx)\, dx = 0$$

for any properly integrable function $f(x)$ on $[0, 1]$. [Sketch $s(nx)$, VIII **3**.]

105. Let $f(x)$ be properly integrable over $[a, b]$. Then we can prove that

$$\lim_{n\to\infty} \int_a^b f(x) \sin nx\, dx = 0.$$

106 (continued). Yet

$$\lim_{n\to\infty} \int_a^b f(x)\, |\sin nx|\, dx = \frac{2}{\pi} \int_a^b f(x)\, dx.$$

Suppose that the function $f(x)$ is bounded on the interval $[a, b]$ and that this interval is subdivided by the points $x_0, x_1, x_2, \ldots, x_{n-1}, x_n$,