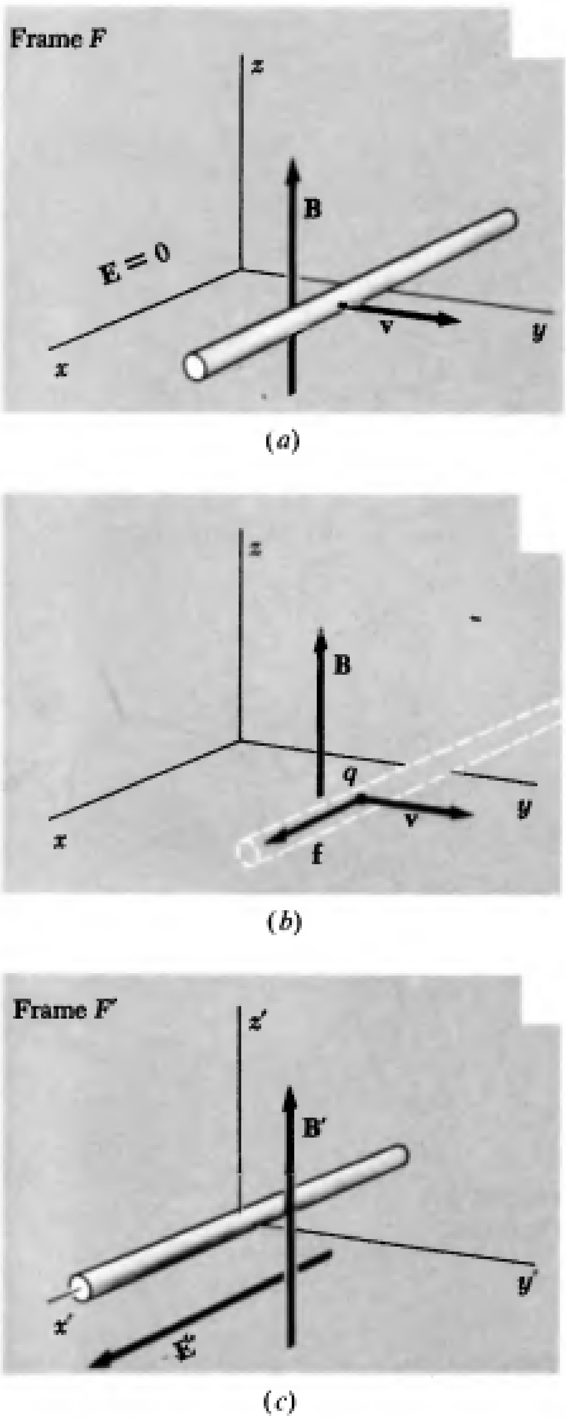
FIGURE 7.2
(*a*) A conducting rod moves through a magnetic field. (*b*) Any charge q that travels with the rod is acted upon by the force $(q/c)\,\mathbf{v} \times \mathbf{B}$. (*c*) The reference frame F' moves with the rod; in this frame there is an electric field $\mathbf{E}'$.



other, and the contact of the inducing one with the battery made when the inductive effect was required; but as the particular action might be supposed to be exerted only at the moments of making and breaking contact, the induction was produced in another way. Several feet of copper wire were stretched in wide zigzag forms, representing the letter W, on one surface of a broad board; a second wire was stretched in precisely similar forms on a second board, so that when brought near the first, the wires should everywhere touch, except that a sheet of thick paper was interposed. One of these wires was connected with the galvanometer, and the other with a voltaic battery. The first wire was then moved towards the second, and as it approached, the needle was deflected. Being then removed, the needle was deflected in the opposite direction. By first making the wires approach and then recede, simultaneously with the vibrations of the needle, the latter soon became very extensive; but when the wires ceased to move from or towards each other, the galvanometer needle soon come to its usual position.

As the wires approximated, the induced current was in the *contrary* direction to the inducing current. As the wires receded, the induced current was in the *same* direction as the inducing current. When the wires remained stationary, there was no induced current.

In this chapter we study the electromagnetic interaction that Faraday explored in those experiments. From our present viewpoint, induction can be seen as a natural consequence of the force on a charge moving in a magnetic field. In a limited sense, we can derive the induction law from what we already know. In following this course we again depart from the historical order of development, but we do so (borrowing Faraday's own words from the end of the passage first quoted) "to give the most concise view of the whole."

A CONDUCTING ROD MOVING THROUGH A UNIFORM MAGNETIC FIELD

7.2 Figure 7.2*a* shows a straight piece of wire, or slender metal rod, supposed to be moving at constant velocity $\mathbf{v}$ in a direction perpendicular to its length. Pervading the space through which the rod moves there is a uniform magnetic field $\mathbf{B}$, constant in time. This could be supplied by a large solenoid enclosing the entire region of the diagram. The reference frame F with coordinates x, y, z is the one in which this solenoid is at rest. In the absence of the rod there is no electric field in that frame, only the uniform magnetic field $\mathbf{B}$.

The rod, being a conductor, contains charged particles that will move if a force is applied to them. Any charged particle that is carried along with the rod, such as the particle of charge q in Fig. 7.2*b*, necessarily moves through the magnetic field $\mathbf{B}$ and does therefore experience a force

$$\mathbf{f} = \frac{q}{c}\mathbf{v} \times \mathbf{B} \tag{1}$$

With $\mathbf{B}$ and $\mathbf{v}$ directed as shown in Fig. 7.2, the force is in the positive x direction if q is a positive charge, and in the opposite direction for the negatively charged electrons that are in fact the mobile charge carriers in most conductors. The consequences will be the same, whether negatives or positives, or both, are mobile.

When the rod is moving at constant speed and things have settled down to a steady state, the force $\mathbf{f}$ given by Eq. 1 must be balanced, at every point inside the rod, by an equal and opposite force. This can only arise from an electric field in the rod. The electric field develops in this way: the force $\mathbf{f}$ pushes negative charges toward one end of the rod, leaving the other end positively charged. This goes on until these separated charges themselves cause an electric field $\mathbf{E}$ such that, everywhere in the interior of the rod,

$$q\mathbf{E} = -\mathbf{f} \tag{2}$$

Then the motion of charge relative to the rod ceases. This charge distribution causes an electric field outside the rod, as well as inside. The field outside looks something like that of separated positive and negative charges, with the difference that the charges are not concentrated entirely at the ends of the rod but are distributed along it. The external field is sketched in Fig. 7.3*a*. Figure 7.3*b* is an enlarged view of the positively charged end of the rod, showing the charge distribution on the surface and some field lines both outside and inside the conductor. That is the way things look, at any instant of time, in frame F.

Let us observe this system from a frame F' that moves with the rod. Ignoring the rod for the moment, we see in this frame F', indicated in Fig. 7.2*c*, a magnetic field $\mathbf{B}'$ (not much different from $\mathbf{B}$ if v is small) together with a uniform electric field, as given by Eq. 6.63,

$$\mathbf{E}' = -\frac{\mathbf{v}'}{c} \times \mathbf{B}' = \frac{\mathbf{v}}{c} \times \mathbf{B}' \tag{3}$$

When we add the rod to this system, all we are doing is putting a stationary conducting rod into a uniform electric field. There will be a redistribution of charge on the surface of the rod so as to make the electric field zero inside, as in the case of the metal box of Fig. 3.6, or of any other conductor in an electric field. The presence of the magnetic field $\mathbf{B}'$ has no influence on this static charge distribution. Figure

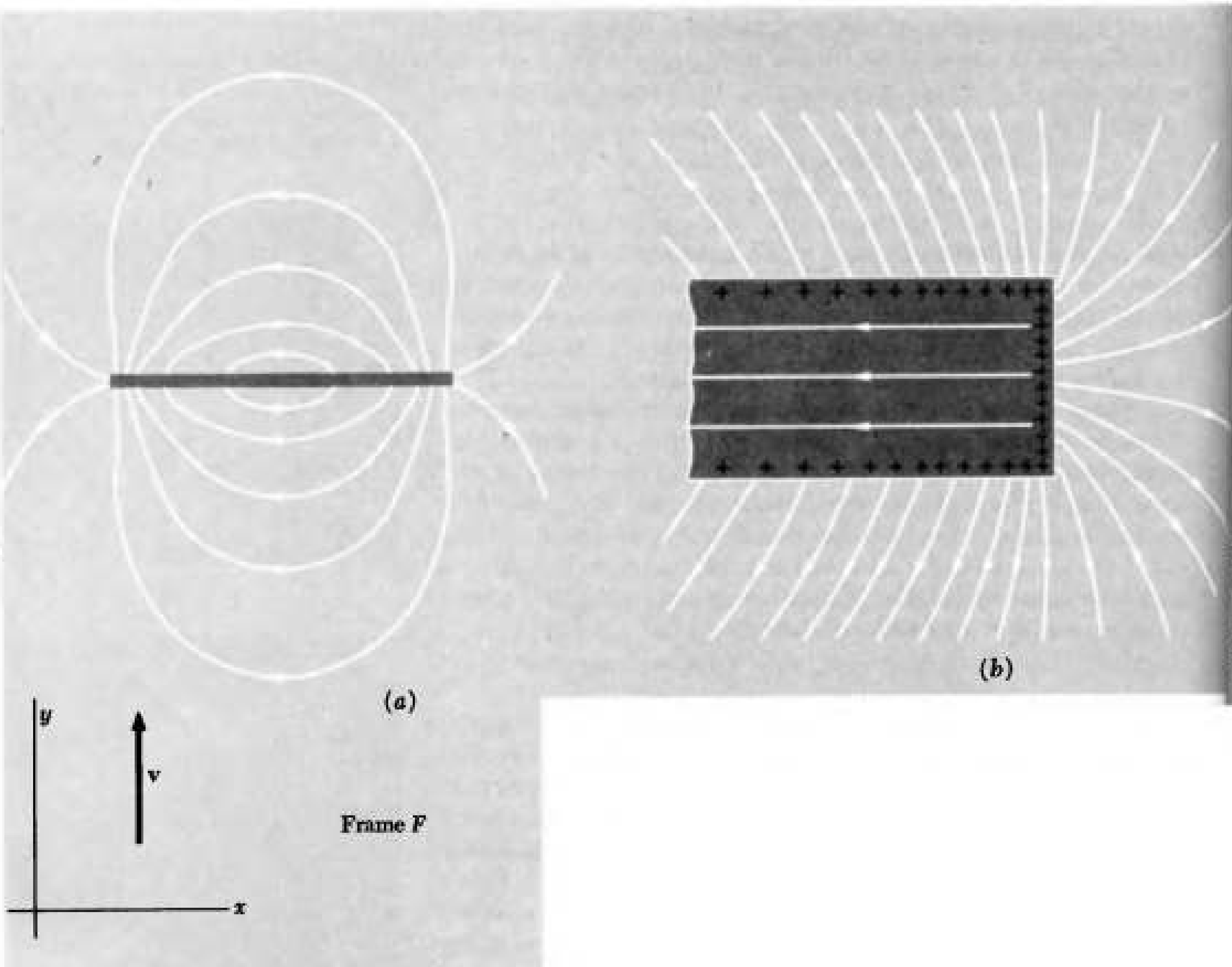


FIGURE 7.3
(a) The electric field, as seen at one instant of time, in the frame F. There is an electric field in the vicinity of the rod, and also inside the rod. The sources of the field are charges on the surface of the rod, as shown in (b), the enlarged view of the right-hand end of the rod.

7.4a shows some electric field lines in the frame F', and in the magnified view of the end of the rod in Fig. 7.4b we observe that the electric field *inside* the rod is zero.

Except for the Lorentz contraction, which is second order in v/c, the charge distribution seen at one instant in frame F, Fig. 7.3b, is the same as that seen in F'. The electric fields differ because the field in Fig. 7.3 is that of the surface charge distribution alone, while the electric field we see in Fig. 7.4 is the field of the surface charge distribution *plus* the uniform electric field that exists in that frame of reference. An observer in F says: "Inside the rod there has developed an electric field $\mathbf{E} = (\mathbf{v}/c) \times \mathbf{B}$, exerting a force $q\mathbf{E} = -q(\mathbf{v}/c) \times \mathbf{B}$ which just balances the force $q(\mathbf{v}/c) \times \mathbf{B}$ that would otherwise cause any charge q to move along the rod." An observer in F' says:

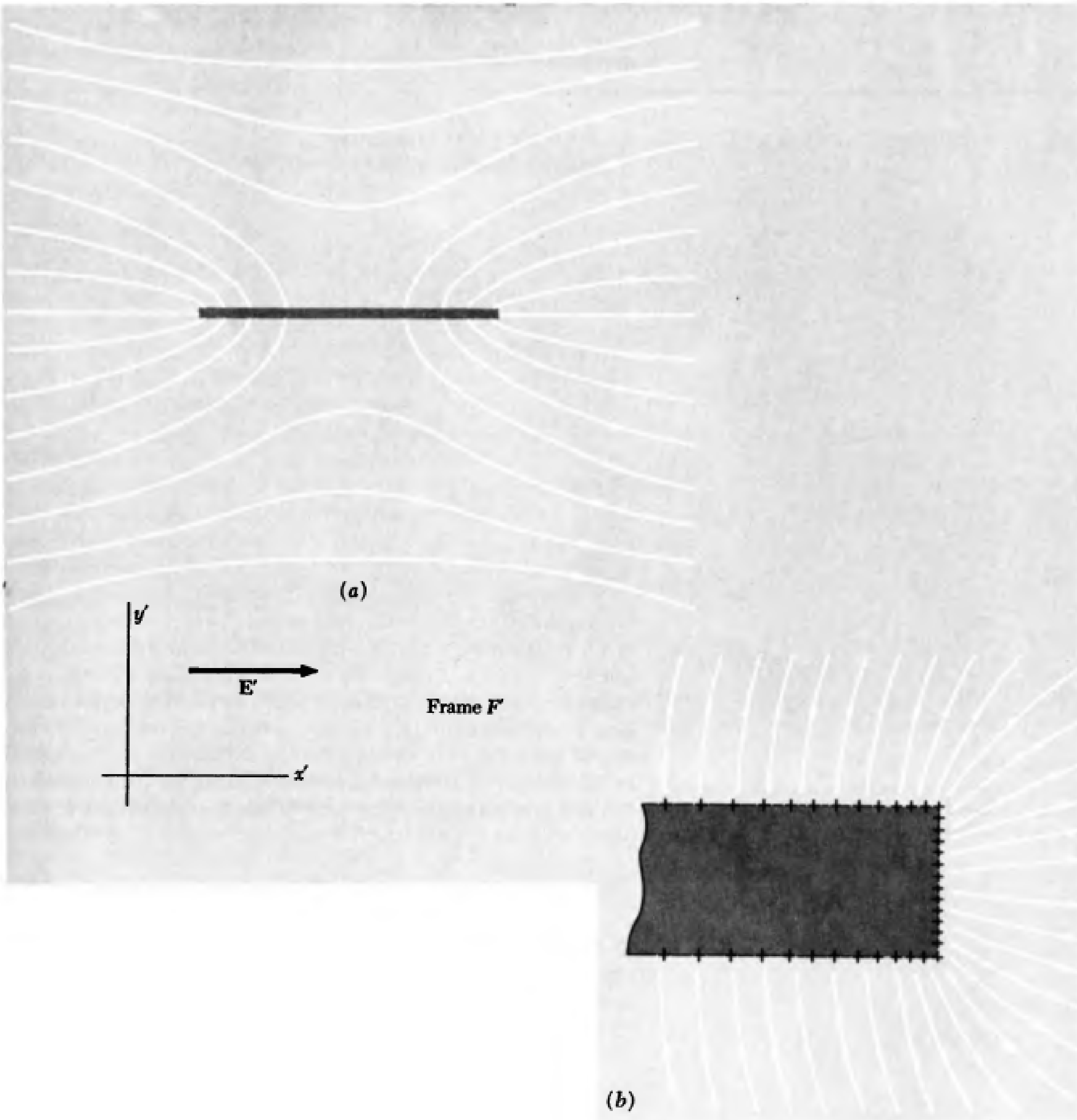


FIGURE 7.4
(*a*) The electric field in the frame F' in which the rod is at rest. This field is a superposition of a general field $\mathbf{E}'$, uniform throughout space, and the field of the surface charge distribution. The result is zero electric field inside the rod, shown in magnified detail in (*b*). Compare with Fig. 7.3.

"Inside the rod there is no electric field, and although there is a uniform magnetic field here, no force arises from it because no charges are moving." Each account is correct.

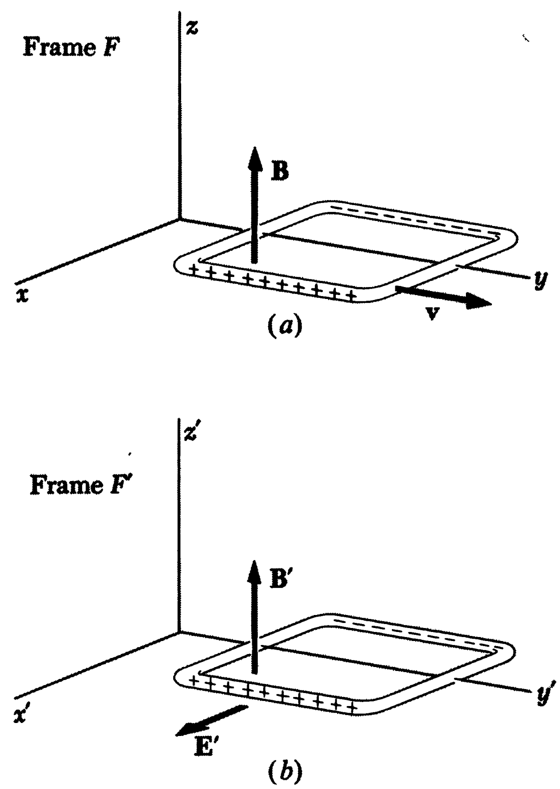


FIGURE 7.5
(*a*) Here the wire loop is moving in a uniform magnetic field **B**. (*b*) Observed in the frame F', in which the loop is at rest, the fields are $\mathbf{B}'$ and $\mathbf{E}'$.

A LOOP MOVING THROUGH A NONUNIFORM MAGNETIC FIELD

7.3 What if we made a rectangular loop of wire, as shown in Fig. 7.5, and moved it at constant speed through the uniform field **B**? To predict what will happen, we need only ask ourselves—adopting the frame F'—what would happen if we put such a loop into a uniform electric field. Obviously two opposite sides of the rectangle would acquire some charge, but that would be all. Suppose, however, that the field **B** in the frame F, though constant in time, is *not uniform* in space. To make this vivid, we show in Fig. 7.6 the field **B** with a short solenoid as its source. This solenoid, together with the battery that supplies its constant current, is fixed near the origin in the frame F. (We said earlier there is no electric field in F; if we really use a solenoid of finite resistance to provide the field, there will be an electric field associated with the battery and this circuit. It is irrelevant to our problem and can be ignored. Or we can pack the whole solenoid, with its battery, inside a metal box, making sure the total charge is zero.)

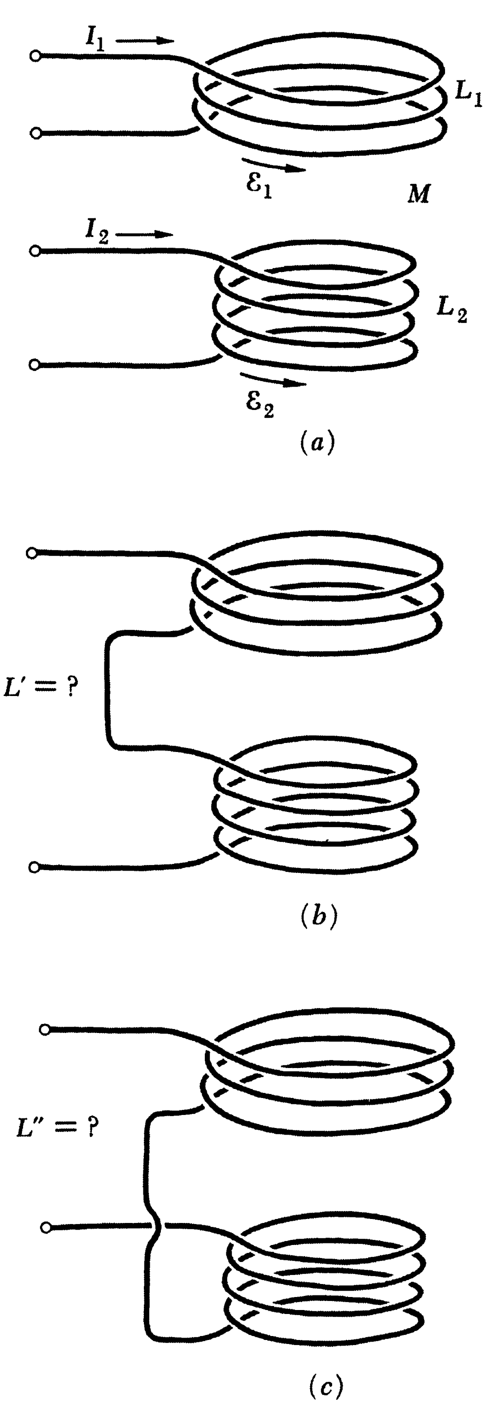
Now, with the loop moving with speed v in the y direction, in the frame F, let its position at some instant t be such that the magnetic field strength is B_1 at the left side of the loop and B_2 along the right side (Fig. 7.6). Let **f** denote the force which acts on a charge q that rides along with the loop. This force is a function of position on the loop, at this instant of time. Let's evaluate the line integral of **f**, taken around the whole loop: On the two sides of the loop which lie parallel to the direction of motion, **f** is perpendicular to the path element $d\mathbf{s}$, so these give nothing. Taking account of the contributions from the other two sides, each of length w, we have

$$\int \mathbf{f} \cdot d\mathbf{s} = \frac{qv}{c}(B_1 - B_2)w \tag{4}$$

If we imagine a charge q to move all around the loop, in a time short enough so that the position of the loop has not changed appreciably, then Eq. 4 gives the work done by the force **f.** The work done *per unit charge* is $(1/q)\int \mathbf{f} \cdot d\mathbf{s}$. We call this quantity *electromotive force*. We use the symbol $\mathcal{E}$ for it, and often shorten the name to emf. $\mathcal{E}$ has the same dimensions as electric potential. It is measured in statvolts, or ergs per unit charge, in the CGS system. The SI unit is the volt.

$$\mathcal{E} = \frac{1}{q}\int \mathbf{f} \cdot d\mathbf{s} \tag{5}$$

The term *electromotive force* was introduced earlier, in Section 4.10. It was defined as the work per unit charge involved in moving a charge around a circuit containing a voltaic cell. We now broaden the defi-



PROBLEM 7.11

wire each, wound around a large block of wood. The turns of the second spiral (that is, single-layer coil) were interposed between those of the first, but separated from them by twine. The diameter of the copper wire itself was $\frac{1}{20}$ inch. He does not give the dimensions of the wooden block or the number of turns in the coils. In the experiment, one of these coils was connected to a "battery of 100 plates." See if you can make a rough estimate of the duration in seconds and magnitude in amperes of the pulse of current that passed through his galvanometer.

7.11 Part (*a*) of the figure shows two coils with self-inductances L_1 and L_2. In the relative position shown their mutual inductance is M. The positive current direction and the positive electromotive force direction in each coil are defined by the arrows in the figure. The equations relating currents and electromotive forces are

$$\mathcal{E}_1 = -L_1 \frac{dI_1}{dt} \underset{?}{\pm} M \frac{dI_2}{dt} \qquad \text{and} \qquad \mathcal{E}_2 = -L_2 \frac{dI_2}{dt} \underset{?}{\pm} M \frac{dI_1}{dt}$$

Given that M is always to be taken as a positive constant, how must the signs be chosen in these equations? What if we had chosen, as we might have, the other direction for positive current, and for positive electromotive force, in the lower coil? Now connect the two coils together as in part (*b*) of the figure to form a single circuit. What is the inductance L' of this circuit, expressed in terms of L_1, L_2, and M? What is the inductance L'' of the circuit formed by connecting the coils as shown in (*c*)? Which circuit, (*b*) or (*c*), has the greater self-inductance? Considering that the self-inductance of any circuit must be a positive quantity (why couldn't it be negative?), see if you can draw a general conclusion, valid for any conceivable pair of coils, concerning the relative magnitude of L_1, L_2, and M.

7.12 An ocean current flows at a speed of 2 knots (approximately 1 meter/sec) in a region where the vertical component of the earth's magnetic field is 0.35 gauss. The conductivity of seawater in that region is 0.04 (ohm-cm)$^{-1}$. On the assumption that there is no other horizontal component of **E** than the motional term $(\mathbf{v}/c) \times \mathbf{B}$, find the density of horizontal electric current in amps/m^2. If you were to carry a bottle of seawater through the earth's field at this speed, would such a current be flowing in it?

7.13 A coil with resistance of 0.01 ohm and self-inductance 0.50 millihenry is connected across a large 12-volt battery of negligible internal resistance. How long after the switch is closed will the current reach 90 percent of its final value? At that time, how much energy, in joules, is stored in the magnetic field? How much energy has been withdrawn from the battery up to that time?

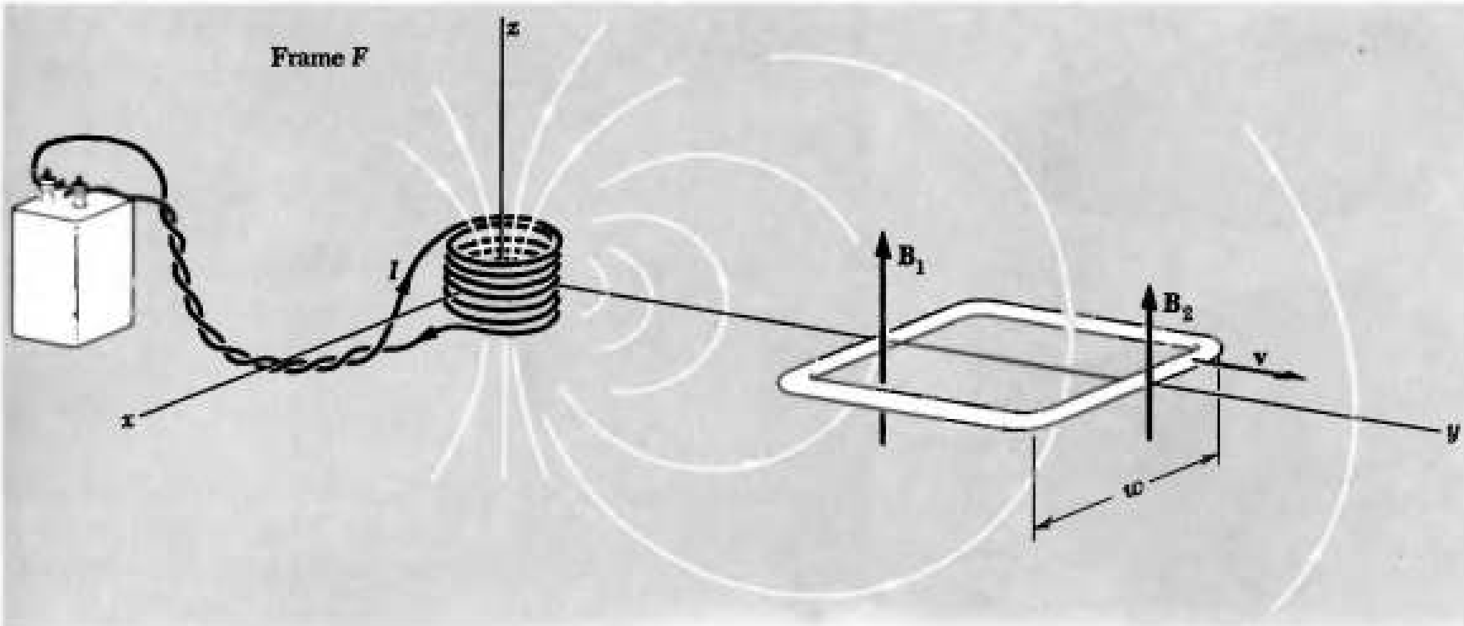


FIGURE 7.6
Here the field **B**, observed in F, is not uniform. It varies in both direction and magnitude from place to place.

nition of emf to include any influence that causes charge to circulate around a closed path. If the path happens to be a physical circuit with resistance R, then the emf $\mathcal{E}$ will cause a current to flow according to Ohm's law: $I = \mathcal{E}/R$. In the particular case we are considering, **f** is the force that acts on a charge moving in a magnetic field, and $\mathcal{E}$ has the magnitude

$$\mathcal{E} = \frac{vw}{c}(B_1 - B_2) \tag{6}$$

The electromotive force given by Eq. 6 is related in a very simple way to the *rate of change of magnetic flux* through the loop. By the magnetic flux through a loop we mean the surface integral of **B** over a surface which has the loop for its boundary. The flux Φ through the closed curve or loop C in Fig. 7.7*a* is given by the surface integral of **B** over S_1:

$$\Phi_{S_1} = \int_{S_1} \mathbf{B} \cdot d\mathbf{a}_1 \tag{7}$$

We could draw infinitely many surfaces bounded by C. Figure 7.7*b* shows another one, S_2. Why don't we have to specify which surface to use in computing the flux? It *doesn't make any difference* because $\int \mathbf{B} \cdot d\mathbf{a}$ will have the same value for all surfaces. Let's take a minute to settle this point once and for all. The flux through S_2 will

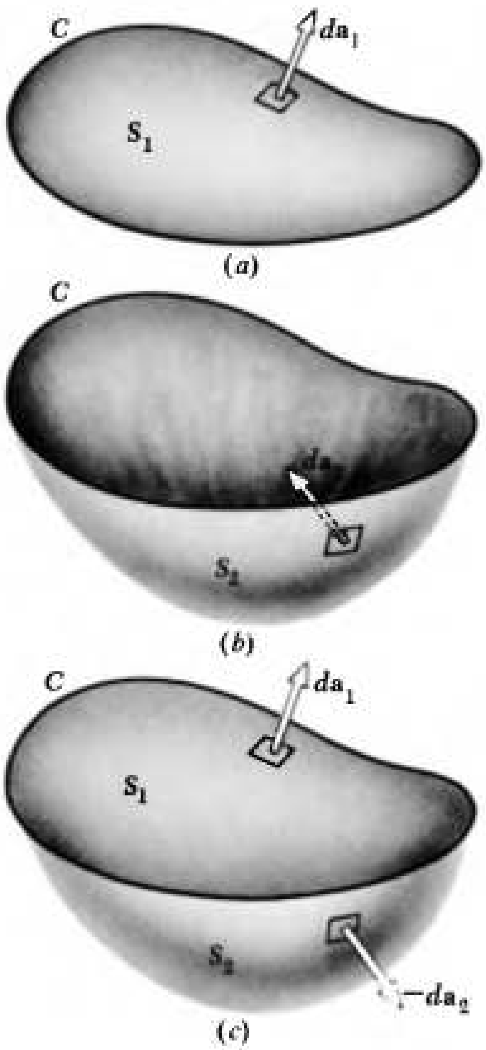


FIGURE 7.7
(a) The flux through C is
$\Phi = \int_{S_1} \mathbf{B} \cdot d\mathbf{a}_1$
(b) S_2 is another surface which has C as its boundary. This will do just as well for computing Φ.
(c) Combining S_1 and S_2 to make a closed surface, for which $\int \mathbf{B} \cdot d\mathbf{a}$ must vanish, proves that $\int_{S_1} \mathbf{B} \cdot d\mathbf{a}_1 = \int_{S_2} \mathbf{B} \cdot d\mathbf{a}_2$.

be $\int_{S2} \mathbf{B} \cdot d\mathbf{a}_2$. Notice that we let the vector $d\mathbf{a}_2$ stick out from the upper side of S_2, to be consistent with our choice of side of S_1. This will give a positive number if the net flux through C is upward.

$$\Phi_{S_2} = \int_{S_2} \mathbf{B} \cdot d\mathbf{a}_2 \tag{8}$$

We learned in Section 6.2 that the magnetic field has zero divergence: div $\mathbf{B} = 0$. It follows then from Gauss' theorem that, if S is any *closed* surface ("balloon") and V is the volume inside it:

$$\int_S \mathbf{B} \cdot d\mathbf{a} = \int_V \text{div}\, \mathbf{B}\, dv = 0 \tag{9}$$

Apply this to the closed surface, rather like a kettledrum, formed by joining our S_1 to S_2, as in Fig. 7.7c. On S_2 the outward normal is *opposite* the vector $d\mathbf{a}_2$ we used in calculating the flux through C. Thus

$$0 = \int_S \mathbf{B} \cdot d\mathbf{a} = \int \mathbf{B} \cdot d\mathbf{a}_1 + \int \mathbf{B} \cdot (-d\mathbf{a}_2)$$

or

$$\int_{S_1} \mathbf{B} \cdot d\mathbf{a}_1 = \int_{S_2} \mathbf{B} \cdot d\mathbf{a}_2 \tag{10}$$

This shows that it doesn't matter which surface we use to compute the flux through C.

This is all pretty obvious if you realize that div $\mathbf{B} = 0$ implies a kind of spatial conservation of flux. As much flux enters any volume as leaves it. (We are considering the situation in the whole space at one instant of time.) It is often helpful to visualize "tubes" of flux. A flux tube (Fig. 7.8) is a surface at every point on which the magnetic field line lies in the plane of the surface. It is a surface through which no flux passes, and we can think of it as containing a certain amount of flux, as a telephone cable contains wires. Through any closed curve drawn tightly around a flux tube, the same flux passes. This could be said about the electric field $\mathbf{E}$ only for regions where there is no electric charge, since div $\mathbf{E} = 4\pi\rho$. The magnetic field always has zero divergence everywhere.

Returning now to the moving rectangular loop, let us find the *rate of change* of flux through the loop. In time dt the loop moves a distance $v\, dt$. This changes in two ways the total flux through the loop, which is $\int \mathbf{B} \cdot d\mathbf{a}$ over a surface spanning the loop. As you can see in Fig. 7.9, flux is gained at the right, in amount $B_2 wv\, dt$, while an

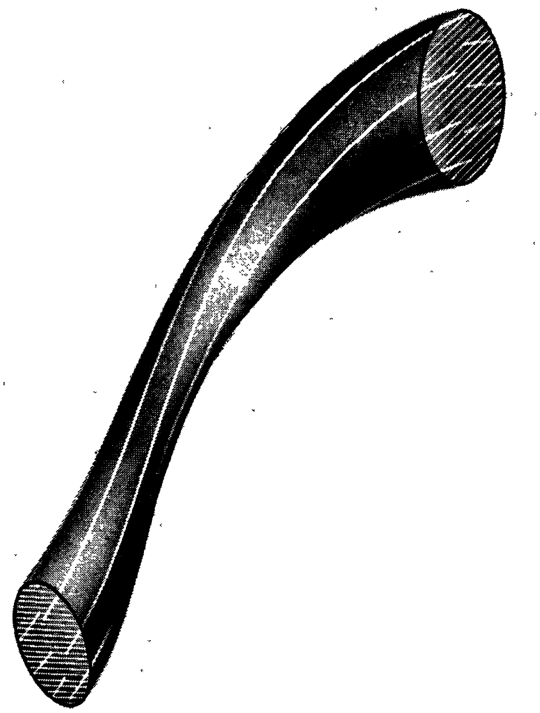


FIGURE 7.8
A flux tube. Magnetic field lines lie in the surface of the tube. The tube encloses a certain amount of flux Φ. No matter where you chop it, you will find that $\int \mathbf{B} \cdot d\mathbf{a}$ over the section has this same value Φ. A flux tube doesn't have to be round. You can start somewhere with any cross section, and the course of the field lines will determine how the section changes size and shape as you go along the tube.

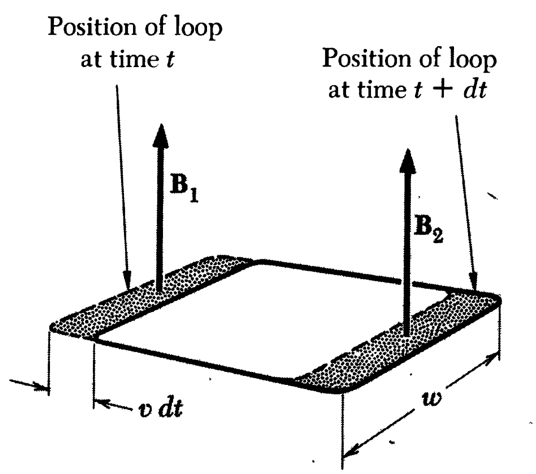


FIGURE 7.9
In the interval dt the loop gains an increment of flux $B_2wv\,dt$ and loses an increment $B_1wv\,dt$.

amount of flux $B_1wv\,dt$ is lost at the left. Hence $d\Phi$, the change in flux through the loop in time dt, is

$$d\Phi = -(B_1 - B_2)wv\,dt \tag{11}$$

Comparing Eq. 11 with Eq. 6, we see that, in this case at least, the electromotive force can be expressed as

$$\mathcal{E} = -\frac{1}{c}\frac{d\Phi}{dt} \tag{12}$$

We can show that this holds quite generally, for a loop of any shape moving in any manner. The loop C in Fig. 7.10 occupies the position C_1 at time t, and it is moving so that it occupies the position C_2 at time $t + dt$. A particular element of the loop $d\mathbf{s}$ has been transported with velocity $\mathbf{v}$ to its new position. S indicates a surface that spans the loop at time t. The flux through the loop at this instant of time is

$$\Phi(t) = \int_S \mathbf{B} \cdot d\mathbf{a} \tag{13}$$

The magnetic field $\mathbf{B}$ comes from sources that are stationary in our frame of reference and remains constant in time, at any point fixed in this frame. At time $t + dt$ a surface which spans the loop is the original surface S, left fixed in space, augmented by the "rim" dS. (Remember, we are allowed to use *any* surface spanning the loop to compute the flux through it.) Thus

FIGURE 7.10
The loop moves from position C_1 to position C_2 in time dt.

$$\Phi(t + dt) = \int_{S+dS} \mathbf{B} \cdot d\mathbf{a} = \Phi(t) + \int_{dS} \mathbf{B} \cdot d\mathbf{a} \tag{14}$$

Hence the change in flux, in time dt, is just the flux through the rim dS, $\int_{dS} \mathbf{B} \cdot d\mathbf{a}$. On the rim, an element of surface area $d\mathbf{a}$ can be expressed as $(\mathbf{v}\, dt) \times d\mathbf{s}$, so the integral over the surface dS can be written as an integral around the path C, in this way:

$$d\Phi = \int_{dS} \mathbf{B} \cdot d\mathbf{a} = \int_C \mathbf{B} \cdot [(\mathbf{v}\, dt) \times d\mathbf{s}] \tag{15}$$

Since dt is a constant for the integration, we can factor it out and have

$$\frac{d\Phi}{dt} = \int_C \mathbf{B} \cdot (\mathbf{v} \times d\mathbf{s}) \tag{16}$$

The product $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ of any three vectors satisfies the relation $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = -(\mathbf{b} \times \mathbf{a}) \cdot \mathbf{c}$. Using this identity to rearrange the integrand in Eq. 16, we have

$$\frac{d\Phi}{dt} = -\int_C (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{s} \tag{17}$$

Now the force on a charge q which is carried along by the loop is just $q(\mathbf{v} \times \mathbf{B})/c$, so the electromotive force, which is the line integral around the loop of the force per unit charge, is just

$$\mathcal{E} = \frac{1}{c} \int_C (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{s} \tag{18}$$

Comparing Eq. 17 with Eq. 18 we get the simple relation already given in Eq. 12, but valid now for arbitrary shape and motion of the loop. (We did not even have to assume that $\mathbf{v}$ is the same for all parts of the loop!) In summary, the line integral around a moving loop of $\mathbf{f}/q$, the force per unit charge, is just $-1/c$ times the rate of change of flux through the loop.

The sense of the line integral and the direction in which flux is called positive are to be related by a right-hand-thread rule. For instance, in Fig. 7.6, the flux is *upward* through the loop and is *decreasing*. Taking the minus sign in Eq. 12 into account, our rule would predict an electromotive force which would tend to drive a positive charge around the loop in a counterclockwise direction, as seen looking down on the loop (Fig. 7.11).

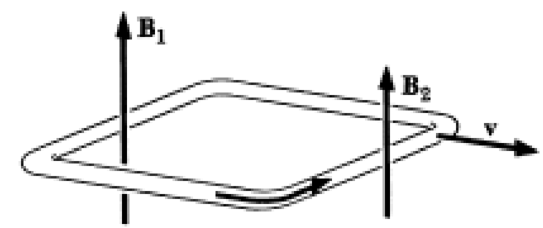
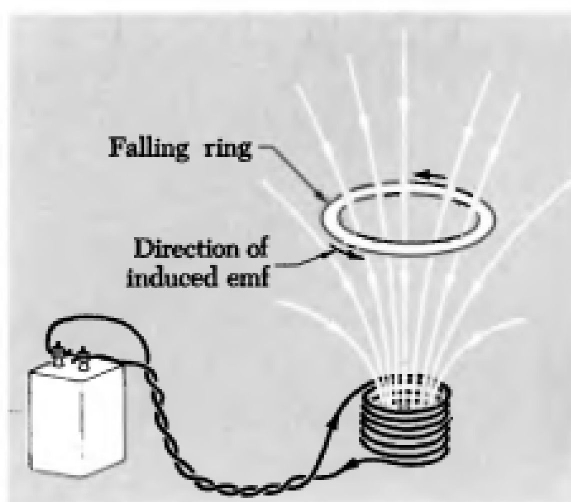


FIGURE 7.11
The flux through the loop is upward and is decreasing in magnitude as time goes on. The arrow shows the direction of the electromotive force, that is, the direction in which positive charge tends to be driven.

There is a better way to look at this question of sign and direction. Notice that if a current should flow in the direction of the induced electromotive force, in the situation shown in Fig. 7.11, this current itself would create some flux through the loop in a direction to *counteract* the assumed flux change. That is an essential physical fact, and not the consequence of an arbitrary convention about signs and directions. It is a manifestation of the tendency of systems to resist change. In this context it is traditionally called *Lenz's law*.

Another example of Lenz's law is illustrated in Fig. 7.12. The conducting ring is falling in the magnetic field of the coil. The flux through the ring is *downward* and is *increasing* in magnitude. To counteract this change, some new flux upward is needed. It would take a current flowing around the ring in the direction of the arrows to produce such flux. Lenz's law assures us that the induced emf will be in the right direction to cause such a current.

FIGURE 7.12
As the ring falls, the downward flux through the ring is increasing. Lenz's law tells us that the induced emf will be in the direction indicated by the arrows, for that is the direction in which current must flow to produce upward flux through the ring. The system reacts so as to oppose the change that is occurring.



If the electromotive force causes current to flow in the loop which is shown in Figs. 7.6 and 7.11, as it will if the loop has a finite resistance, some energy will be dissipated in the wire. What supplies this energy? To answer that, consider the force that acts on the current in the loop if it flows in the sense indicated by the arrow in Fig. 7.11. The conductor on the right, in the field B_2, will experience a force toward the right, while the opposite side of the loop, in the field B_1, will be pushed toward the left. But B_1 is greater than B_2, so the net force on the loop is toward the left, *opposing the motion*. To keep the loop moving at constant speed some external agency has to do work, and the energy thus invested eventually shows up as heat in the wire. Imagine what would happen if Lenz's law were violated, or if the force on the loop were to act in a direction to assist the motion of the loop!

A very common element in electrical machinery and electrical instruments is a loop or coil that rotates in a magnetic field. Let's apply what we have just learned to the system shown in Fig. 7.13, a single loop rotating at constant speed in a magnetic field that is approximately uniform. The mechanical essentials, shaft, bearings, drive, etc., are not drawn. The field $\mathbf{B}$ is provided by the two fixed coils.

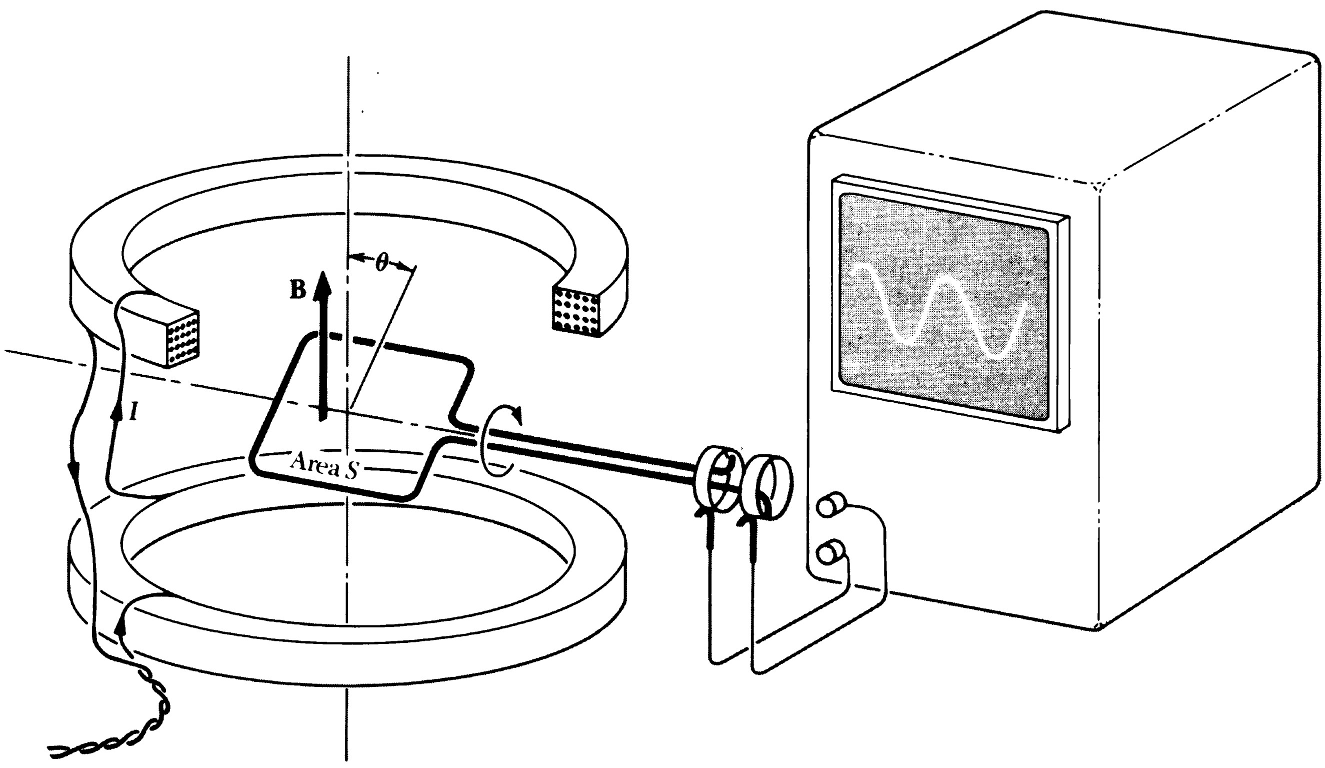


FIGURE 7.13
The two coils produce a magnetic field **B** which is approximately uniform in the vicinity of the loop. In the loop, rotating with angular velocity ω, a sinusoidally varying electromotive force is induced.

Suppose the loop rotates with angular velocity ω. in radians/sec. If its position at any instant is specified by the angle θ, then $\theta = \omega t + \alpha$, where the constant α is simply the position of the loop at $t = 0$. The component of **B** perpendicular to the plane of the loop is $B \sin \theta$. Therefore the flux through the loop at time t is

$$\Phi(t) = SB \sin(\omega t + \alpha) \tag{19}$$

where S is the area of the loop. For the induced electromotive force we then have

$$\mathcal{E} = -\frac{1}{c}\frac{d\Phi}{dt} = -\frac{SB\omega}{c}\cos(\omega t + \alpha) \tag{20}$$

If the loop instead of being closed is connected through slip rings to external wires, as shown in Fig. 7.13, we can detect at these terminals a sinusoidally alternating potential difference.

A numerical example will show how the units work out. Suppose the area of the loop in Fig. 7.13 is 80 cm^2, the field strength B is 50 gauss, and the loop is rotating at 30 revolutions per sec. Then $\omega = 2\pi$

$\times$ 30, or 188 radians/sec. The amplitude, that is, the maximum magnitude of the oscillating electromotive force induced in the loop, is

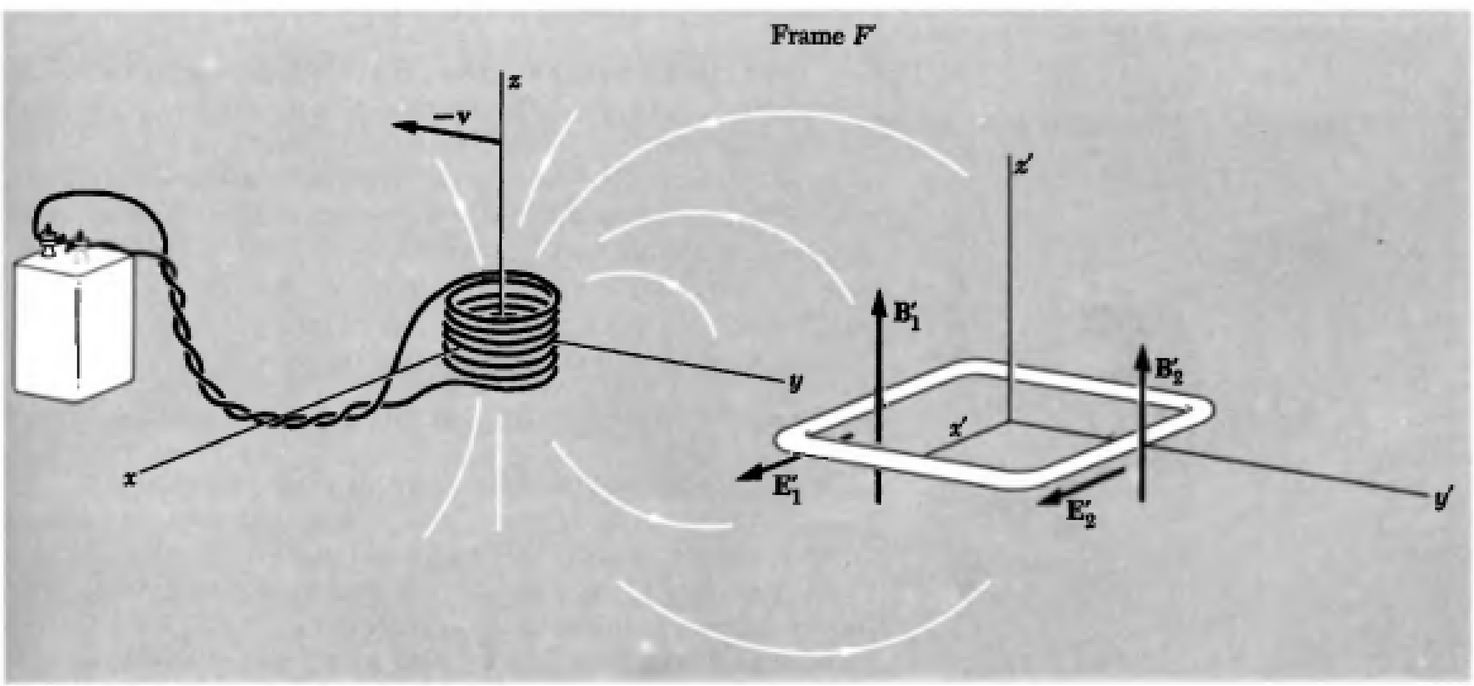
$$\begin{aligned}\mathcal{E}_0 = \frac{SB\omega}{c} &= \frac{(80\ \text{cm}^2)(50\ \text{gauss})(188\ \text{sec}^{-1})}{3 \times 10^{10}\ \text{cm/sec}} \\ &= 2.51 \times 10^{-5}\ \text{gauss-cm or statvolt}\end{aligned} \tag{21}$$

One gauss-cm is equivalent to 1 statvolt. Remember that electric field E and magnetic field B have the same dimensions in our CGS system, being related by a dimensionless factor v/c.

A STATIONARY LOOP WITH THE FIELD SOURCE MOVING

7.4 We can, if we like, look at the events depicted in Fig. 7.6 from a frame of reference that is moving with the loop. That can't change the physics, only the words we use to describe it. Let F', with coordinates x', y', z', be the frame attached to the loop, which we now regard as stationary (Fig. 7.14). The coil and battery, stationary in frame F, are moving in the $-y'$ direction with velocity $\mathbf{v}' = -\mathbf{v}$. Let B_1' and B_2' be the magnetic field measured at the two ends of the loop by

FIGURE 7.14
As observed in the frame F', the loop is at rest, the field source is moving. The fields $\mathbf{B}'$ and $\mathbf{E}'$ are both present and are functions of both position and time.



observers in F' at some instant t'. At these positions there will be an electric field in F'. Equation 6.63 tells us that

$$\begin{aligned} \mathbf{E}_1' &= -\frac{\mathbf{v}' \times \mathbf{B}_1'}{c} = \frac{\mathbf{v} \times \mathbf{B}_1'}{c} \\ \mathbf{E}_2' &= -\frac{\mathbf{v}' \times \mathbf{B}_2'}{c} = \frac{\mathbf{v} \times \mathbf{B}_2'}{c} \end{aligned} \tag{22}$$

For observers in F' this is a genuine electric field. It is not an electrostatic field. The line integral of $\mathbf{E}'$ around any closed path in F' is not generally zero. In fact, the line integral of $\mathbf{E}'$ around the rectangular loop is

$$\int \mathbf{E}' \cdot d\mathbf{s}' = \frac{wv}{c}(B_1' - B_2') \tag{23}$$

We can call the line integral in Eq. 23 the electromotive force $\mathscr{E}'$ on this path. If a charged particle moves once around the path, $\mathscr{E}'$ is the work done on it, per unit charge. $\mathscr{E}'$ is related to the rate of change of flux through the loop. To see this, note that, while the loop itself is stationary, the *magnetic field pattern* is now moving with the velocity $-\mathbf{v}$ of the source. Hence for the flux lost or gained at either end of the loop, in a time interval dt', we get a result similar to Eq. 11, and we conclude that

$$\mathscr{E}' = -\frac{1}{c}\frac{d\Phi'}{dt'} \tag{24}$$

We can summarize as follows the descriptions in the two frames of reference, F, in which the source of $\mathbf{B}$ is at rest, and F', in which the loop is at rest:

An observer in F says, "We have here a magnetic field which, though it is not uniform spatially, is constant in time. There is no electric field. That wire loop over there is moving with velocity $\mathbf{v}$ through the magnetic field, so the charges in it are acted on by a force $(\mathbf{v}/c) \times \mathbf{B}$ dynes per unit charge. The line integral of this force per unit charge, taken around the whole loop, is the electromotive force $\mathscr{E}$ and it is equal to $-(1/c)(d\Phi/dt)$. The flux Φ is $\int \mathbf{B} \cdot d\mathbf{a}$ over a surface S which, at some instant of time t by my clock, spans the loop."

An observer in F' says, "This loop is stationary, and only an electric field could cause the charges in it to move. But there is in fact an electric field $\mathbf{E}'$. It seems to be caused by that magnetlike object which happens at this moment to be whizzing by with a velocity $-\mathbf{v}$, producing at the same time a rather strong magnetic field $\mathbf{B}'$. The electric field is such that $\int \mathbf{E}' \cdot d\mathbf{s}'$ around this stationary loop is not zero but instead is equal to $-1/c$ times the rate of change of flux through the loop, $d\Phi'/dt'$. The flux Φ' is $\int \mathbf{B}' \cdot d\mathbf{a}'$ over a surface spanning the loop, the values of B' to be measured all over this surface at some one instant t', by my clock."

Our conclusions so far are relativistically exact. They hold for any speed $v \leqq c$ provided we observe scrupulously the distinctions between $\mathbf{B}$ and $\mathbf{B}'$, t and t', etc. If $v \ll c$, so that v^2/c^2 can be neglected, $\mathbf{B}'$ will be practically equal to $\mathbf{B}$, and we can safely ignore also the distinction between t and t'.

A UNIVERSAL LAW OF INDUCTION

7.5 Let's carry out three experiments with the apparatus shown in Fig. 7.15. The tables are on wheels so that they can be easily moved. A sensitive galvanometer has been connected to our old rectangular loop, and to increase any induced electromotive force we put several turns of wire in the loop rather than one. Frankly though, our sensitivity might still be marginal, with the feeble source of magnetic field pictured. Perhaps you can devise a more practical version of the experiment in the laboratory.

Experiment I With constant current in the coil and table 1 stationary, table 2 moves toward the right with speed v. The *galvanometer deflects*. We are not surprised; we have already analyzed this situation in Section 7.3.

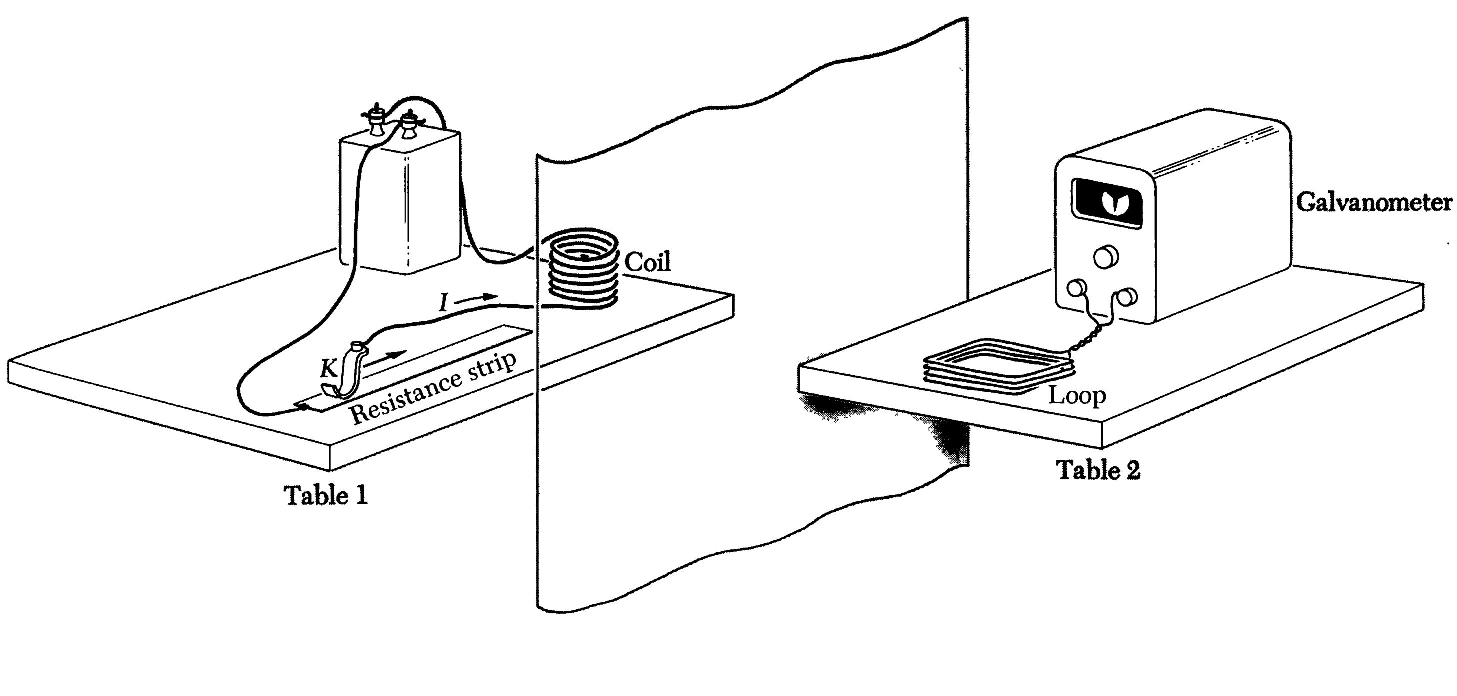


FIGURE 7.15
We imagine that either table can move or, with both tables fixed, the current I in the coil can be gradually changed.

Experiment II With constant current in the coil and table 2 stationary, table 1 moves to the left with speed v. The *galvanometer deflects.* This doesn't surprise us either. We have just discussed the equivalence of Experiments I and II, an equivalence which is an example of Lorentz invariance or, for the low speeds of our tables, Galilean invariance. We know that in both experiments the deflection of the galvanometer can be related to the rate of change of flux of $\mathbf{B}$ through the loop.

Experiment III Both tables remain at rest, but we vary the current I in the coil by sliding the contact K along the resistance strip. We do this in such a way that the *rate of decrease* of the field $\mathbf{B}$ at the loop is the same as it was in Experiments I and II. *Does the galvanometer deflect?*

For an observer stationed at the loop on table 2 and measuring the magnetic field in that neighborhood as a function of time and position, there is no way to distinguish among Experiments I, II, and III. Imagine a black cloth curtain between the two tables. Although there might be minor differences between the field configurations for II and III, an observer who did not know what was behind the curtain could not decide, on the basis of local $\mathbf{B}$ measurements alone, which case it was. Therefore if the galvanometer did *not* respond with the same deflection in Experiment III, it would mean that the relation between the magnetic and electric fields in a region depends on the nature of a remote source. Two magnetic fields essentially similar in their local properties would have associated in one case, but not in the other, an electric field with $\int \mathbf{E} \cdot d\mathbf{s} \neq 0$.

We find by experiment that III *is* equivalent to I and II. The galvanometer deflects, by the same amount as before. Faraday's experiments were the first to demonstrate this fundamental fact. The electromotive force we observe depends only on the rate of change of the flux of $\mathbf{B}$, and not on anything else. We can state as a universal relation *Faraday's law of induction:*

> If C is some closed curve, stationary in coordinates x, y, z, if S is a surface spanning C, and if $\mathbf{B}(x, y, z, t)$ is the magnetic field measured in x, y, z, at any time t, then
>
> $$\mathcal{E} = \int_C \mathbf{E} \cdot d\mathbf{s} = -\frac{1}{c}\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{a} = -\frac{1}{c}\frac{d\Phi}{dt} \qquad (25)$$

Using the vector derivative curl, we can express this law in differential form. If the relation

$$\int_C \mathbf{E} \cdot d\mathbf{s} = -\frac{1}{c}\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{a} \tag{26}$$

is true for *any* curve C and spanning surface S, as our law asserts, it follows that at any point

$$\text{curl } \mathbf{E} = -\frac{1}{c}\frac{d\mathbf{B}}{dt} \tag{27}$$

To show that Eq. 27 follows from Eq. 26, we proceed as usual to let C shrink down around a point, which we take to be a nonsingular point for the function $\mathbf{B}$. Then in the limit the variation of $\mathbf{B}$ over the small patch of surface $\mathbf{a}$ that spans C will be negligible and the surface integral will approach simply $\mathbf{B} \cdot \mathbf{a}$. Now by definition (Eq. 2.61) the limit approached by $\int_C \mathbf{E} \cdot d\mathbf{s}$ as the patch shrinks is $\mathbf{a} \cdot \text{curl } \mathbf{E}$. Thus we have, in the limit,

$$\mathbf{a} \cdot \text{curl } \mathbf{E} = -\frac{1}{c}\frac{d}{dt}(\mathbf{B} \cdot \mathbf{a}) = \mathbf{a} \cdot \left(-\frac{1}{c}\frac{d\mathbf{B}}{dt}\right) \tag{28}$$

Since this holds for *any* infinitesimal $\mathbf{a}$, it must be that†

$$\text{curl } \mathbf{E} = -\frac{1}{c}\frac{d\mathbf{B}}{dt} \tag{29}$$

Recognizing that $\mathbf{B}$ may depend on position as well as time we shall write $\partial\mathbf{B}/\partial t$ in place of $d\mathbf{B}/dt$. We have then these two entirely equivalent statements of the law of induction:

$$\boxed{\begin{aligned} \int_C \mathbf{E} \cdot d\mathbf{s} &= -\frac{1}{c}\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{a} \\ \text{curl } \mathbf{E} &= -\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t} \end{aligned}} \tag{30}$$

In Eq. 30 the electric field $\mathbf{E}$ is to be expressed in our CGS units of statvolts/cm, with $\mathbf{B}$ in gauss, $d\mathbf{s}$ in cm, $d\mathbf{a}$ in cm^2, and c in cm/sec. The electromotive force $\mathcal{E} = \int_C \mathbf{E} \cdot d\mathbf{s}$ will then be given in statvolts.

†If that isn't obvious, note that choosing $\mathbf{a}$ in the x direction will establish that $(\text{curl } \mathbf{E})_x = -\frac{1}{c}\frac{dB_x}{dt}$, and so on.