in the inertial frame F', will gradually increase from zero. However, as we are concerned with the instantaneous acceleration, only infinitesimal values of v' are involved anyway, and the restriction on Eq. 14 is rigorously fulfilled. For $E_\perp$, the transverse field component in F, the transformation is $E'_\perp = \gamma E_\perp$, so that $dp'_\perp/dt' = qE'_\perp = q\gamma E_\perp$. But on transforming the force back to frame F we have $dp_\perp/dt = (1/\gamma)(dp'_\perp/dt')$, so the γ drops out after all:

$$\frac{dp_\perp}{dt} = \frac{1}{\gamma}(\gamma E_\perp q) = qE_\perp \tag{17}$$

The message of Eqs. 16 and 17 is simply this: The force on a charged particle in motion through F is q times the electric field **E** in that frame, *strictly independent* of the velocity of the particle. Figure 5.19 is a reminder of this fact, and of the way we discovered it.

You have already used this result earlier in the course, where you were simply told that the contribution of the electric field to the force on a moving charge is q**E**. Because this is familiar and so simple, you may think it is obvious and we have been wasting our time proving it. Now we could have taken it as an empirical fact. It has been verified over an enormous range, up to velocities so close to the speed of light, in the case of electrons, that the factor γ is 10^4. From that point of view it is a most remarkable law. Our discussion in this chapter has shown that this fact is also a direct consequence of charge invariance.

INTERACTION BETWEEN A MOVING CHARGE AND OTHER MOVING CHARGES

5.9 We know that there can be a velocity-dependent force on a moving charge. That force is associated with a *magnetic field,* the sources of which are electric currents, that is, other charges in motion. Oersted's experiment showed that electric currents could influence magnets, but at that time the nature of a magnet was totally mysterious. Soon Ampère and others unraveled the interaction of electric currents with each other, as in the attraction observed between two parallel wires carrying current in the same direction. This led Ampère to the hypothesis that a magnetic substance contains permanently circulating electric currents. If so, Oersted's experiment could be understood as the interaction of the galvanic current in the wire with the permanent microscopic currents which gave the compass needle its special properties. Ampère gave a complete and elegant mathematical formulation of the interaction of steady currents, and of the equivalence of magnetized matter to systems of permanent currents. His brilliant conjecture about the actual nature of magnetism in iron had to wait a century, more or less, for its ultimate confirmation.

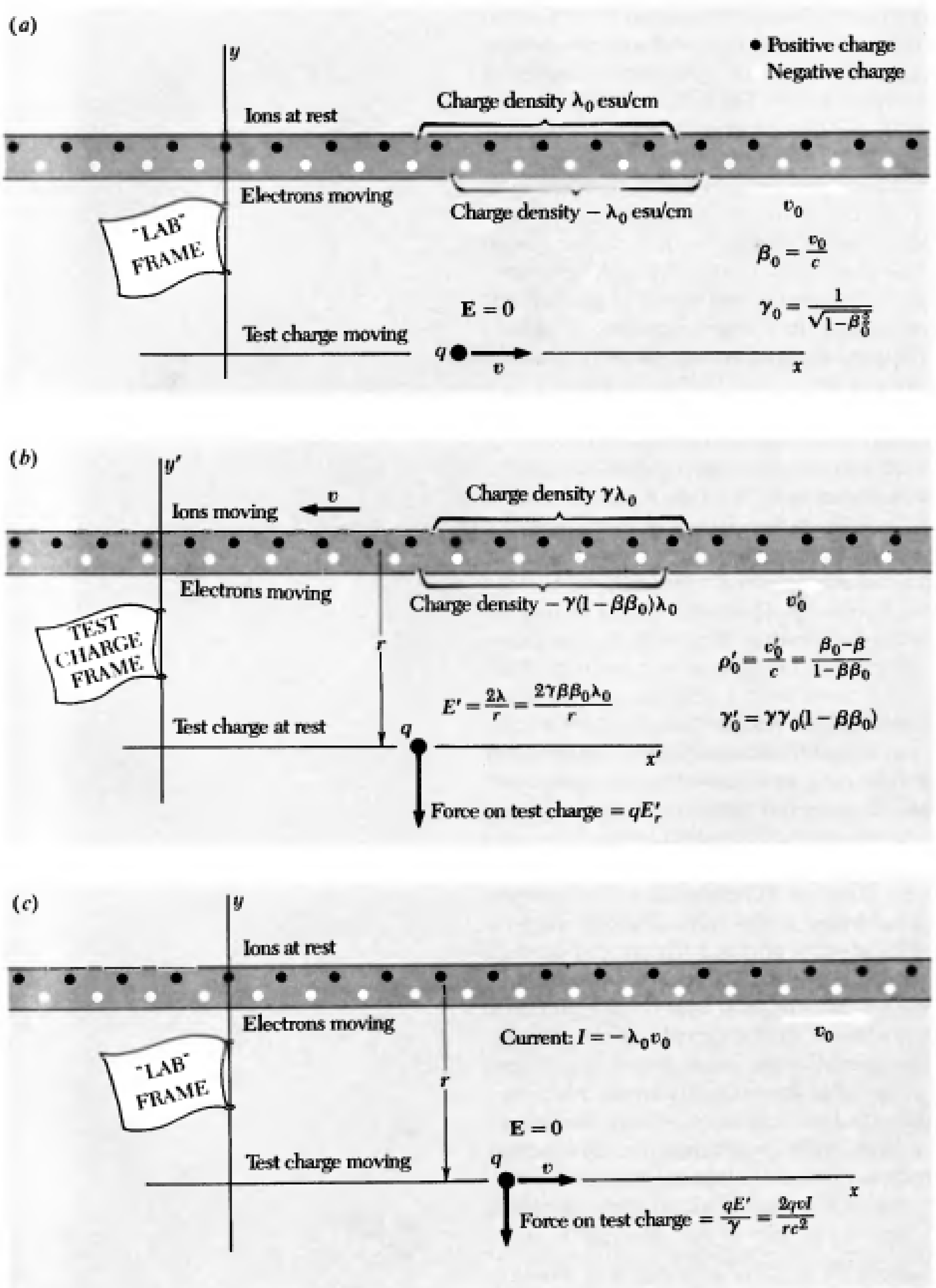
Whether the magnetic manifestations of electric currents arose from anything *more* than the simple transport of charge was not clear to Ampère and his contemporaries. Would the motion of an electrostatically charged object cause effects like those produced by a continuous galvanic current? Later in the century Maxwell's theoretical work suggested the answer should be *yes*. The first direct evidence was obtained by Henry Rowland, to whose experiment we shall return at the end of Chapter 6.

From our present vantage point, the magnetic interaction of electric currents can be recognized as an inevitable corollary to Coulomb's law. If the postulates of relativity are valid, if electric charge is invariant, and if Coulomb's law holds, then, as we shall now show, the effects we commonly call "magnetic" are bound to occur. They will emerge as soon as we examine the electric interaction between a moving charge and other moving charges. A very simple system will illustrate this.

In the lab frame of Fig. 5.20*a*, with spatial coordinates x, y, z, there is a line of positive charges, at rest and extending to infinity in both directions. We shall call them ions for short. Indeed, they might represent the copper ions that constitute the solid substance of a copper wire. There is also a line of negative charges that we shall call electrons. These are all moving to the right with speed v_0. In a real wire the electrons would be intermingled with the ions; we've separated them in the diagram for clarity. The linear density of positive charge is λ_0 in esu/cm. It happens that the linear density of negative charge along the line of electrons is exactly equal in magnitude. That is, any given length of "wire" contains at a given instant the same number of electrons and protons.† The net charge on the wire is zero. Gauss' law tells us there can be no flux from a cylinder that contains no charge, so the electric field must be zero everywhere outside the wire. A test charge q at rest near this wire experiences no force whatever.

Suppose the test charge is not at rest in the lab frame but is moving with speed v in the x direction. Transform to a frame moving with the test charge, the x', y' frame in Fig. 5.20*b*. The test charge q is here at rest, but something else has changed: The wire appears to be charged! There are two reasons for that: The positive ions are closer together, and the electrons are farther apart. Because the lab frame in which the positive ions are at rest is moving with speed v, the distance between positive ions as seen in the test charge frame is contracted by $\sqrt{1 - v^2/c^2}$, or $1/\gamma$. The linear density of positive charge in this frame is correspondingly greater; it must be $\gamma\lambda_0$. The density of negative charge takes a little longer to calculate, for the electrons were already moving with speed v_0 in the lab frame. Hence their linear density in the lab frame, which was $-\lambda_0$, had already been increased

†It doesn't have to, but that equality can always be established, if we choose, by adjusting the number of electrons per unit length. We assume that has been done.


(a)
y
• Positive charge
Negative charge
Charge density λ_0 esu/cm
Ions at rest
Electrons moving
Charge density $-\lambda_0$ esu/cm
v_0
"LAB" FRAME
$\beta_0 = \frac{v_0}{c}$
$\mathbf{E} = 0$
$\gamma_0 = \frac{1}{\sqrt{1-\beta_0^2}}$
Test charge moving
q
v
x
(b)
y′
v
Ions moving
Charge density $\gamma\lambda_0$
Electrons moving
Charge density $-\gamma(1-\beta\beta_0)\lambda_0$
v_0'
TEST CHARGE FRAME
r
$\beta_0' = \frac{v_0'}{c} = \frac{\beta_0 - \beta}{1-\beta\beta_0}$
$E' = \frac{2\lambda}{r} = \frac{2\gamma\beta\beta_0\lambda_0}{r}$
$\gamma_0' = \gamma\gamma_0(1-\beta\beta_0)$
Test charge at rest
q
x′
Force on test charge $= qE_r'$
(c)
y
Ions at rest
Electrons moving
Current: $I = -\lambda_0 v_0$
v_0
"LAB" FRAME
r
$\mathbf{E} = 0$
Test charge moving
q
v
x
Force on test charge $= \frac{qE'}{\gamma} = \frac{2qvI}{rc^2}$

by a Lorentz contraction. In the electrons' own rest frame the negative charge density must have been $-\lambda_0/\gamma_0$, where γ_0 is the *Lorentz factor* that goes with v_0.

Now we need the speed of the electrons in the test charge frame in order to calculate their density there. To find that velocity (v_0' in Fig. 5.20*b*) we must add the velocity $-v$ to the velocity v_0, remembering to use the relativistic formula for the addition of velocities (Eq. 6 in Appendix A). Let $\beta_0' = v_0'/c$, $\beta_0 = v_0/c$, and $\beta = v/c$. Then

$$\beta_0' = \frac{\beta_0 - \beta}{1 - \beta\beta_0} \tag{18}$$

The corresponding Lorentz factor γ_0', obtained from Eq. 18 with a little algebra, is

$$\gamma_0' = (1 - \beta_0'^2)^{-1/2} = \gamma\gamma_0(1 - \beta\beta_0) \tag{19}$$

This is the factor by which the linear density of negative charge in the electrons' own rest frame is enhanced when it is measured in the test charge frame. The total linear density of charge in the wire in the test charge frame, λ', can now be calculated:

$$\lambda' = \gamma\lambda_0 - \frac{\lambda_0}{\gamma_0}\underbrace{\gamma\gamma_0(1 - \beta\beta_0)} = \gamma\beta\beta_0\lambda_0 \tag{20}$$

factor for transformation to test charge frame

positive charge density in ions' rest frame

negative charge density in electrons' rest frame

factor for transformation to test charge frame

The wire is positively charged. Gauss's law guarantees the existence of a radial electric field E_r' given by our familiar formula for the field of any infinite line charge:

$$E_r' = \frac{2\lambda'}{r'} = \frac{2\gamma\beta\beta_0\lambda_0}{r'} \tag{21}$$

At the location of the test charge q this field is in the $-y'$ direction. The test charge will experience a force

$$F_y' = qE_y' = -\frac{2q\gamma\beta\beta_0\lambda_0}{r'} \tag{22}$$

Now let's return to the lab frame, pictured again in Fig. 5.20*c*. What is the magnitude of the force on the charge q as measured there? If its value is qE_y' in the rest frame of the test charge, observers in the lab frame will report a force smaller by the factor $1/\gamma$. Since $r = r'$, the force on our moving test charge, measured in the lab frame, is

FIGURE 5.20
A test charge q moving parallel to a current in a wire. (*a*) In the lab frame the wire, in which the positive charges are fixed, is at rest. The current consists of electrons moving to the right with speed v_0. The net charge on the wire is zero. There is no electric field outside the wire. (*b*) In a frame in which the test charge is at rest the positive ions are moving to the left with speed v and the electrons are moving to the right with speed v_0'. The linear density of a positive charge is greater than the linear density of negative charge. The wire appears positively charged, with an external field E_r' which causes a force qE_r' on the stationary test charge q. (*c*) That force transformed back to the lab frame has the magnitude qE_r'/γ, which is proportional to the product of the speed v of the test charge and the current in the wire, $-\lambda_0 v_0$.

$$F_y = \frac{F'_y}{\gamma} = -\frac{2q\beta\beta_0\lambda_0}{r} \qquad (23)$$

Now $-\lambda_0 v_0$ or $-\lambda_0\beta_0 c$ is just the total current I in the wire, in the lab frame, for it is the amount of charge flowing past a given point per second. We'll call current positive if it is equivalent to positive charge flowing in the positive x direction. Our current in this example is negative. Our result can be written this way:

$$F_y = \frac{2I}{rc^2} q v_x \qquad (24)$$

We have found that in the lab frame the moving test charge experiences a force in the y direction which is proportional to the current in the wire, and to the velocity of the test charge in the x direction.

It is a remarkable fact that the force on the moving test charge does not depend separately on the velocity or density of the charge carriers but only on the product, $\beta_0\lambda_0$ in our example, that determines the charge transport. If we have a certain current I, say 10^7 esu/sec which is the same as 3.3 milliamps, it does not matter whether this current is composed of high-energy electrons moving with 99 percent of the speed of light, of electrons in a metal executing nearly random thermal motions with a slight drift in one direction, or of charged ions in solution with positive ions moving one way, negatives the other. Or it could be any combination of these, as Problem 5.18 will demonstrate. Furthermore, the force on the test charge is strictly proportional to the velocity of the test charge v. Our derivation was in no way restricted to small velocities, either for the charge carriers in the wire or for the moving charge q. Equation 24 is exact, with no restrictions.

Let's see how this explains the mutual repulsion of conductors carrying currents in opposite directions, as shown in Fig. 5.1*b* at the beginning of this chapter. Two such wires are represented in the lab frame in Fig. 5.21*a*. Assume the wires are uncharged in the lab frame. Then there is no electrical force from the opposite wire on the positive ions which are stationary in the lab frame. Transferring to a frame in which one set of electrons is at rest (Fig. 5.21*b*), we find that in the other wire the electron distribution is Lorentz-contracted more than the positive ion distribution. Because of that the electrons at rest in this frame will be repelled by the other wire. But when we transfer to the frame in which those other electrons are at rest (Fig. 5.21*c*), we find the same situation. They too will be repelled. These repulsive forces will be observed in the lab frame as well, modified only by the factor γ. We conclude that the two streams of electrons will repel one another in the lab frame. The stationary positive ions, although they feel no direct electrical force from the other wire, will be the indirect bearers of this repulsive force if the electrons remain confined within the wire. So the wires will be pushed apart, as in Fig. 5.1*b*, until some external force balances the repulsion.

Moving parallel to a current-carrying conductor, the charged particle experienced a force perpendicular to its direction of motion. What if it moves, instead, at right angles to the conductor? A velocity perpendicular to the wire will give rise to a force parallel to the wire—again, a force perpendicular to the particle's direction of motion. To see how this comes about, let us return to the lab frame of that system and give the test charge a velocity v in the y direction, as in Fig. 5.22*a*. Transferring to the rest frame of the test charge (Fig. 5.22*b*), we find the positive ions moving vertically downward. Certainly they cannot cause a horizontal field at the test-charge position. The x' component of the field from an ion on the left will be exactly cancelled by the x' component of the field of a symmetrically positioned ion on the right. The effect we are looking for is caused by the electrons. They are all moving obliquely in this frame, downward and toward the right. Consider the two symmetrically located electrons e_1 and e_2. Their electric fields, relativistically compressed in the direction of the electrons' motion, have been represented by a brush of field lines in the manner of Fig. 5.14. You can see that, although e_1 and e_2 are equally far away

FIGURE 5.21
(*a*) Lab frame with two wires carrying current in opposite directions. As in metal wire, current is due to motion of negative ions (electrons) only. (*b*) Rest frame of electrons in wire 1. Note that in wire 2 positive ions are compressed, but electron distribution is contracted even more. (*c*) Rest frame of electrons in wire 2. Just as in (*b*), the *other* wire appears to these electrons at rest to be negatively charged.

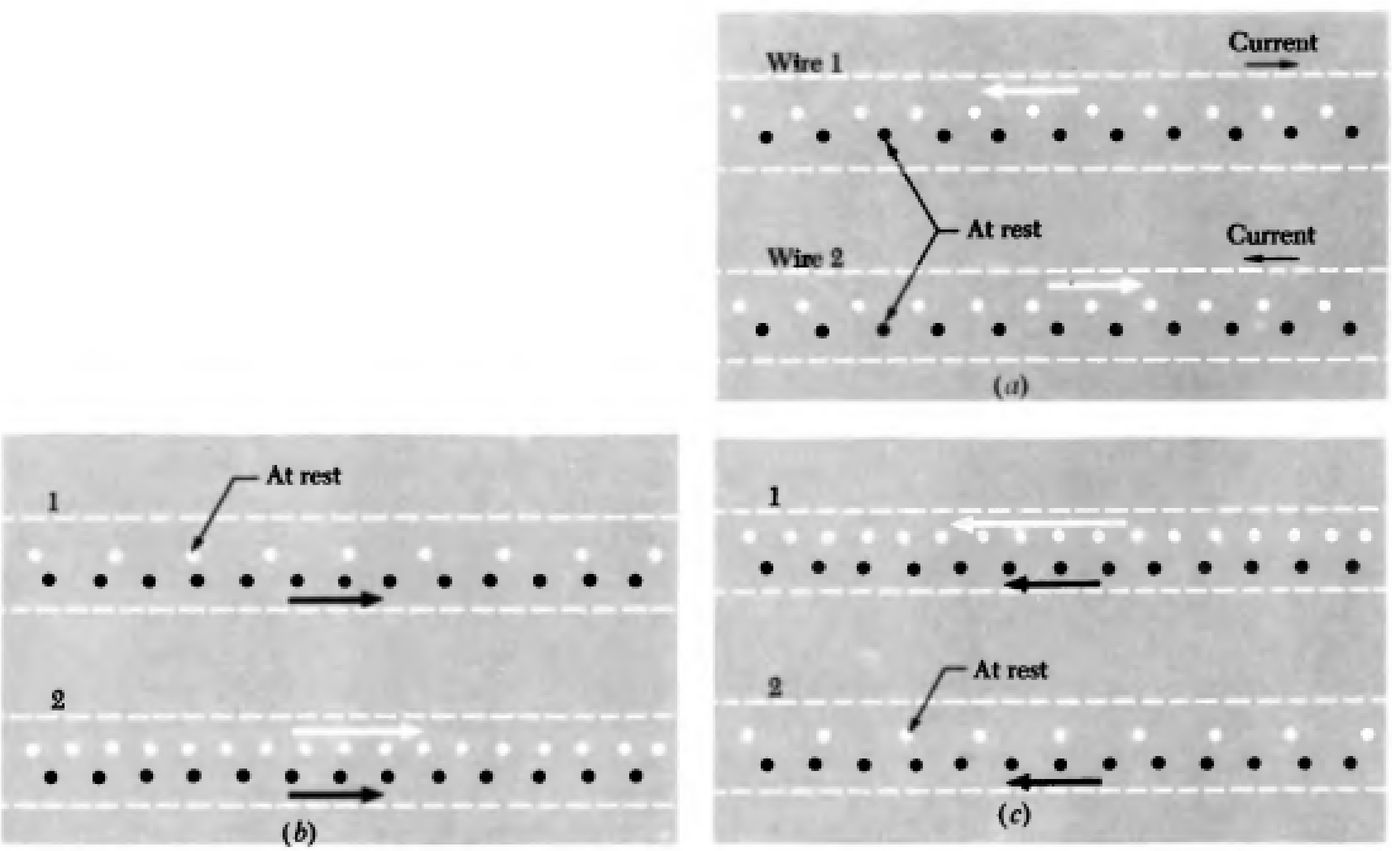
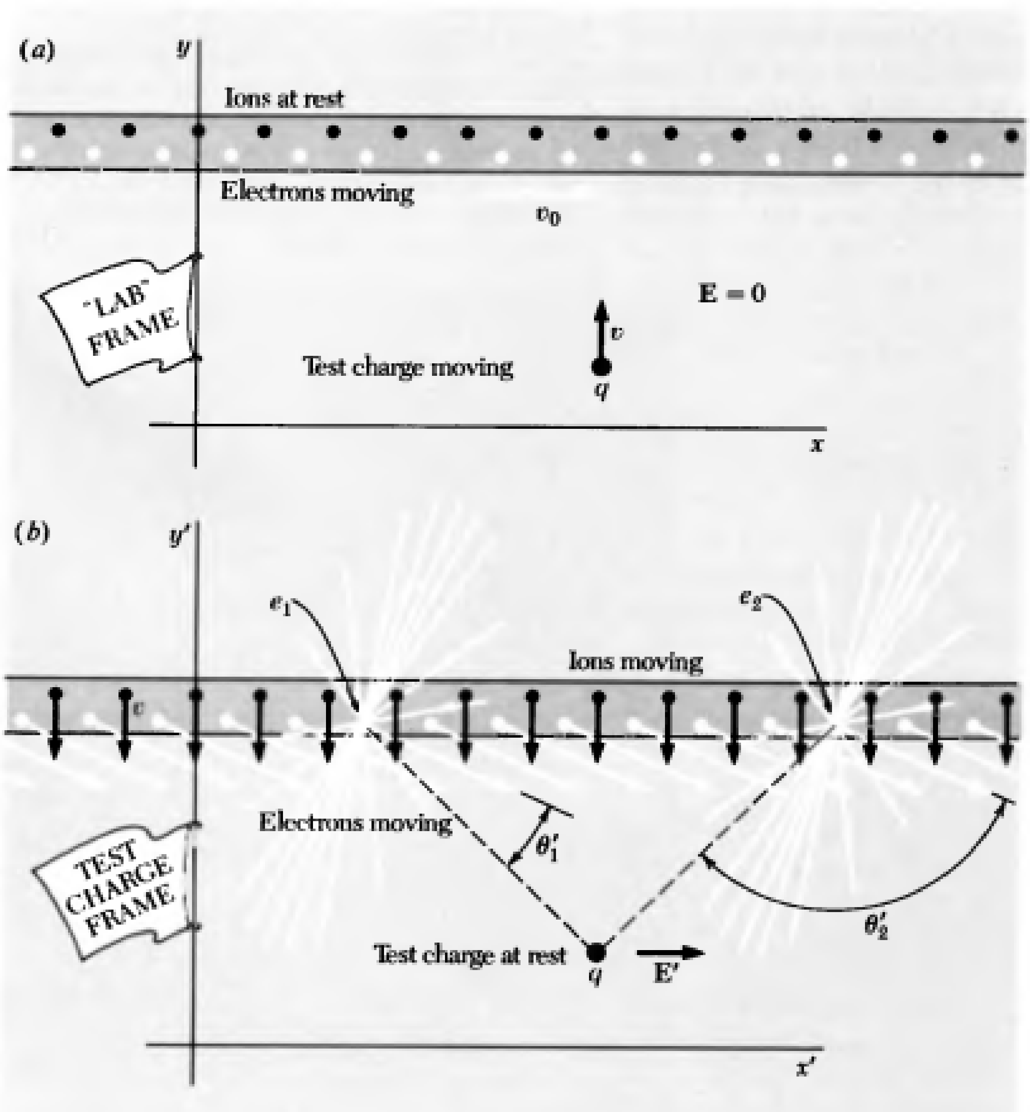


FIGURE 5.22
(*a*) The "wire" with its current of moving negative charges, or "electrons," is the same as in Fig. 5.20, but now the test charge is moving toward the wire. (*b*) In the rest frame of the test charge the positive charges, or "ions," are moving in the $-\hat{\mathbf{y}}$ direction. The electrons are moving obliquely. Because the field of a moving charge is stronger in directions more nearly perpendicular to its velocity, an electron on the right, such as e_2, causes a stronger field at the position of the test charge than does a symmetrically located electron on the left. Therefore the vector sum of the fields has in this frame a component in the $\hat{\mathbf{x}}'$ direction.

from the test charge, the field of electron e_2 will be *stronger* than the field of electron e_1 at that location. That is because the line from e_2 to the test charge is more nearly perpendicular to the direction of motion of e_2. In other words, the angle θ' that appears in the denominator of Eq. 12 is here different for e_1 and e_2, so that $\sin^2\theta_2' > \sin^2\theta_1'$. That will be true for any symmetrically located pair of electrons on the line, as you can verify with the aid of Fig. 5.23. The electron on the right always wins. Summing over all the electrons is therefore bound to yield a resultant field E' in the $\hat{\mathbf{x}}$ direction. The y' component of the electrons' field will be exactly cancelled by the field of the ions. That E_y' is zero is guaranteed by Gauss's law, for the number of charges per unit length of wire is the same as it was in the lab frame. The wire is uncharged in both frames.



The force on our test charge, qE'_x, when transformed back into the lab frame, will be a force proportional to v in the $\hat{\mathbf{x}}$ direction, which is the direction of $\mathbf{v} \times \mathbf{B}$ if $\mathbf{B}$ is a vector in the $\hat{\mathbf{z}}$ direction, pointing at us out of the diagram. We could show that the magnitude of this velocity-dependent force is given here also by Eq. 24: $F = 2qvI/rc^2$. The physics needed is all in Eq. 12, but the integration is somewhat laborious and will not be undertaken here.

In this chapter we have seen how the fact of charge invariance implies forces between electric currents. That does not oblige us to look on one fact as the cause of the other. These are simply two aspects of electromagnetism whose relationship beautifully illustrates the more general law: Physics is the same in all inertial frames of reference.

If we had to analyze every system of moving charges by transforming back and forth among various coordinate systems, our task would grow both tedious and confusing. There is a better way. The overall effect of one current on another, or of a current on a moving charge, can be described completely and concisely by introducing a new field, the magnetic field.

FIGURE 5.23
A closer look at the geometry of Fig. 5.22*b*, showing that, for *any* pair of electrons equidistant from the test charge, the one on the right will have a larger value of $\sin^2\theta'$. Hence, according to Eq. 5.12, it will produce the stronger field at the test charge.