Expect a metric of the form:

$$ds^2 = g_{tt}dt^2 + g_{rr}dr^2 \tag{1}$$

0.1 $\mu = r$ Component

The Lorentz equations for the $\mu = r$ component is:

$$\frac{dp_r}{d\tau} + \Gamma^r_{\alpha\beta} p_\alpha \frac{dx^\beta}{d\tau} = -qF^{r\nu} g_{\nu\beta} \frac{dx^\beta}{d\tau}$$

Thus:

$$\begin{aligned} \frac{d}{d\tau}\frac{dr}{d\tau} + \Gamma^r_{tt}\frac{dt}{d\tau}\frac{dt}{d\tau} + \Gamma^r_{tr}\frac{dt}{d\tau}\frac{dr}{d\tau} + \Gamma^r_{rt}\frac{dr}{d\tau}\frac{dt}{d\tau} + \Gamma^r_{rr}\frac{dr}{d\tau}\frac{dr}{d\tau} \\ = -\frac{q}{m}F^{r\nu}\left(g_{\nu t}\frac{dt}{d\tau} + g_{\nu r}\frac{dr}{d\tau}\right) \end{aligned}$$

using the metric 1:

$$\begin{aligned} \frac{d}{d\tau}\frac{dr}{d\tau} + \Gamma^r_{tt}\frac{dt}{d\tau}\frac{dt}{d\tau} + \Gamma^r_{tr}\frac{dt}{d\tau}\frac{dr}{d\tau} + \Gamma^r_{rt}\frac{dr}{d\tau}\frac{dt}{d\tau} + \Gamma^r_{rr}\frac{dr}{d\tau}\frac{dr}{d\tau} \\ = -\frac{q}{m}F^{rt}\left(g_{tt}\frac{dt}{d\tau}\right) - qF^{rr}\left(g_{rr}\frac{dr}{d\tau}\right) \end{aligned}$$

Since, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, then $F_{rt} = \partial_r A_t - \partial_t A_r = \frac{d\psi}{dr}$ and $F_{rr} = \partial_r A_r - \partial_r A_r = 0$, so:

$$\begin{aligned} \frac{d}{d\tau}\frac{dr}{d\tau} + \Gamma^r_{tt}\frac{dt}{d\tau}\frac{dt}{d\tau} + \Gamma^r_{tr}\frac{dt}{d\tau}\frac{dr}{d\tau} + \Gamma^r_{rt}\frac{dr}{d\tau}\frac{dt}{d\tau} + \Gamma^r_{rr}\frac{dr}{d\tau}\frac{dr}{d\tau} \\ = -\frac{q}{m}g^{tt}g^{rr}\frac{d\psi}{dr}\left(g_{tt}\frac{dt}{d\tau}\right) \end{aligned} \tag{2}$$

Now, the Christoffel symbols we require are $\Gamma^r_{tt}, \Gamma^r_{tr}\left(=\Gamma^r_{rt}\right), \Gamma^r_{rr}$ and are given by:

$$\Gamma^r_{tt} = \frac{1}{2}g^{r\beta}\left(\partial_t g_{\beta t} + \partial_t g_{\beta t} - \partial_\beta g_{tt}\right) = \frac{1}{2}g^{rr}\left(\partial_t g_{rt} + \partial_t g_{tr} - \partial_r g_{tt}\right) = \frac{1}{2}g^{rr}\left(-\partial_r g_{tt}\right)$$

$$\Gamma^r_{tr} = \frac{1}{2}g^{r\beta}\left(\partial_r g_{\beta t} + \partial_t g_{\beta r} - \partial_\beta g_{tr}\right) = \frac{1}{2}g^{rr}\left(\partial_r g_{rt} + \partial_t g_{rr} - \partial_r g_{tr}\right) = 0$$

$$\Gamma^r_{rr} = \frac{1}{2}g^{r\beta}\left(\partial_r g_{\beta r} + \partial_r g_{\beta r} - \partial_\beta g_{rr}\right) = \frac{1}{2}g^{rr}\left(\partial_r g_{rr} + \partial_r g_{rr} - \partial_r g_{rr}\right) = \frac{1}{2}g^{rr}\left(\partial_r g_{rr}\right)$$

Substituting these Christoffel symbols into 2:

$$\frac{d}{d\tau}\frac{dr}{d\tau} + \frac{1}{2}g^{rr}\left(-\partial_r g_{tt}\right)\frac{dt}{d\tau}\frac{dt}{d\tau} + \frac{1}{2}g^{rr}\left(\partial_r g_{rr}\right)\left(\frac{dr}{d\tau}\right)^2 = -\frac{q}{m}g^{tt}g^{rr}\frac{d\psi}{dr}\left(g_{tt}\frac{dt}{d\tau}\right)$$

$$\Rightarrow \frac{d}{d\tau}\frac{dr}{d\tau} - \frac{1}{2}g^{rr}\partial_r g_{tt}\left(\frac{dt}{d\tau}\right)^2 + \frac{1}{2}g^{rr}\partial_r g_{rr}\left(\frac{dr}{d\tau}\right)^2 = -\frac{q}{m}g^{rr}\frac{d\psi}{dr}\frac{dt}{d\tau} \tag{3}$$

The proper time, τ, is given by:

$$d\tau^2 = g_{tt}dt^2 + g_{rr}dr^2$$

Rearranging:

$$1 = g_{tt}\left(\frac{dt}{d\tau}\right)^2 + g_{rr}\left(\frac{dr}{d\tau}\right)^2$$

$$\Rightarrow \left(\frac{dt}{d\tau}\right)^2 = \frac{1}{g_{tt}} - \frac{g_{rr}}{g_{tt}}\left(\frac{dr}{d\tau}\right)^2 \tag{4}$$

Substitute 4 into 3:

$$\frac{d}{d\tau}\frac{dr}{d\tau} - \frac{1}{2}g^{rr}\partial_r g_{tt}\left(\frac{1}{g_{tt}} - \frac{g_{rr}}{g_{tt}}\left(\frac{dr}{d\tau}\right)^2\right) + \frac{1}{2}g^{rr}\partial_r g_{rr}\left(\frac{dr}{d\tau}\right)^2 = -\frac{q}{m}g^{rr}\frac{d\psi}{dr}\frac{dt}{d\tau}$$

$$\Rightarrow \frac{d}{d\tau}\frac{dr}{d\tau} - \frac{g^{rr}\partial_r g_{tt}}{2g_{tt}} + \frac{g^{rr}g_{rr}\partial_r g_{tt}}{2g_{tt}}\left(\frac{dr}{d\tau}\right)^2 + \frac{g^{rr}\partial_r g_{rr}}{2}\left(\frac{dr}{d\tau}\right)^2 = -\frac{q}{m}g^{rr}\frac{d\psi}{dr}\frac{dt}{d\tau}$$

$$\Rightarrow \frac{d}{d\tau}\frac{dr}{d\tau} + \left(\frac{\partial_r g_{tt}}{2g_{tt}} + \frac{\partial_r g_{rr}}{2g_{tt}}\right)\left(\frac{dr}{d\tau}\right)^2 - \frac{\partial_r g_{tt}}{2g_{rr}g_{tt}} = -\frac{q}{m}g^{rr}\frac{d\psi}{dr}\frac{dt}{d\tau}$$