Reciprocity Principle and the Lorentz Transformations

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(Received 7 June 1968)

By using the principle of relativity, together with the customary assumptions concerning the nature of the space–time manifold in special relativity, namely, space–time homogeneity and isotropy of space, a simple but rigorous proof is given of the reciprocity relation for the relative motion of two inertial frames of reference, which is usually assumed as a postulate in the standard derivations of the Lorentz transformations without the principle of invariance of light velocity. A critical discussion is set forth of the question of eliminating the transformations with invariant imaginary velocity, which one unavoidably obtains together with the Lorentz transformations and the Galilean ones in adopting a procedure of this kind.

I. INTRODUCTION

Since the appearance of the classical Einstein paper,[1] in which the foundations of the theory of relativity were first laid down, several other derivations of the Lorentz transformations have been published in the attempt to throw full light on the underlying principles and to clarify both the physical content and the mathematical implications of the latter.[2] In particular, it has been shown as far back as 1911 by Frank and Rothe[2a] that the assumption of the existence of an invariant velocity is not necessary in order to arrive at the correct transformation equations. This is rather a remarkable result, since it shows that the principle of relativity (which establishes the equivalence of all inertial frames of reference in regard to the description of physical phenomena) together with the customary assumptions concerning the nature of the space–time manifold in special relativity, namely, its homogeneity and the isotropy of physical space, point towards the existence of a universal constant which has the meaning of an invariant velocity, so that there is no need to introduce this constant into the theory at the beginning.

Without imposing from the outset the principle of constancy of light velocity, many of the existing standard derivations of the Lorentz transformations make more or less explicit use of the so called *reciprocity principle* which, as is well known, states simply that the velocity of an inertial frame of reference S with respect to another inertial frame of reference S' is the opposite of the velocity of S' with respect to S.[3]

The use of this principle is not strictly necessary to the scope, but it has the advantage of greatly simplifying the derivation of the transformation equations, which would otherwise require rather lengthy calculations and the resort to nonelementary results of the theory of Lie transformation groups.[4] It appears, however, that in the existing literature no sufficiently convincing arguments have been put forward to justify the use of the reciprocity principle. Indeed, it is generally assumed as a justification that the reciprocity relation is a consequence of the principle of relativity, *whereas the latter merely implies the invariance of the relation between direct and reciprocal velocity.*

It is the aim of the present paper to give a simple but rigorous deduction of the reciprocity relation, starting from the three basic postulates of the special theory of relativity, namely, the principle of equivalence of inertial frames, the homogeneity of space–time, and the isotropy of space.[5]

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[1] A. Einstein, Ann. Phys. **17**, 891 (1905).

[2] The existing literature is very wide and rather unrelated and it would be almost impossible to give a fairly complete summary of it. We draw attention to the following references: (a) P. Frank and H. Rothe, Ann. Phys. **34**, 825 (1911); (b) L. A. Pars, Phil. Mag. **42**, 249 (1921); (c) A. S. Eddington, *The Mathematical Theory of Relativity* (Cambridge University Press, London, 1923), Sec. 4; (d) Y. Mimura and T. Iwatsuki, J. Sci. Hiroshima Univ. **A1**, 111 (1931); (e) V. V. Narliker, Proc. Cambridge Phil. Soc. **28**, 460 (1932); (f) G. J. Whitrow, Quart. J. Math. **4**, 161 (1933); (g) L. R. Gomes, Lincei Rend. **21**, 433 (1935); (h) F. Severi, Proc. Phys.-Math. Soc. Japan **18**, 257 (1936); (i) E. Esclangon, Compt. Rend. **202**, 708 (1936); (j) E. Le Roy, *ibid.* **202**, 794 (1936); (k) V. Lalan, *ibid.* **203**, 1491 (1936); Bull. Soc. Math. France **65**, 83 (1937); (l) G. J. Whitrow and E. A. Milne, Z. Astrophys. **15**, 270 (1938); (m) G. Temple, Quart. J. Math. **9**, 283 (1938); (n) H. E. Ives, Proc. Am. Phil. Soc. **95**, 125 (1951); (o) K. D. Stiegler, Compt. Rend. **234**, 1250 (1952); (p) A. W. Ingleton, Nature **171**, 618 (1953); (q) J. Aharoni, *The Special Theory of Relativity* (Oxford University Press, London, 1965), Chap. 1; (r) V. Fock, *The Theory of Space, Time and Gravitation* (Pergamon Press Ltd., London, 1959), Chap. 1 and Appendix A; (s) H. M. Schwartz, Am. J. Phys. **30**, 697 (1962); *Introduction to Special Relativity* (McGraw-Hill Book Co., New York, 1968), Chap. 3; (t) E. C. Zeeman, J. Math. Phys. **5**, 490 (1964); (u) R. Weinstock, Am. J. Phys. **32**, 261 (1964); **33**, 640 (1965); **35**, 892 (1967); (v) V. Mitvalský, *ibid.* **34**, 825 (1966); (w) E. Drake, *ibid.* **34**, 899 (1966); (x) J. L. Strecker, *ibid.* **35**, 13 (1967); (y) L. J. Eisenberg, *ibid.* **35**, 649 (1967); (z) H. Almström, J. Phys. A (Proc. Phys. Soc.) **1**, 331 (1968); (aa) Ya. P. Terletskii, *Paradoxes in the Theory of Relativity* (Plenum Press, Inc., New York, 1968), Chap. 2.

[3] See, for example, Refs. 2(h–j, q, s, u–w, z).

[4] See, for example, Refs. 2(a, k).

[5] A critical analysis of the literature quoted in Footnote 2 and a general discussion concerning the axiomatic derivation of the extended inhomogeneous Lorentz group is the subject of a forthcoming paper.

Once the reciprocity relation has been established, the transformation equations can easily be deduced, as is well known, by making use of their group property, which follows from the principle of relativity.[6] Nevertheless, it is not superfluous to present this deduction again here, since this gives us an opportunity to discuss critically the arguments which have been put forward in favor of excluding the Lorentz transformations with imaginary invariant velocity, which one unavoidably obtains together with the Galilei and the ordinary Lorentz transformations by following a procedure of this kind.

We confine ourselves throughout this paper to the consideration of a two-dimensional space–time and to transformations which conserve the space–time origin. This implies no loss of generality, since any transformation can always be reduced to a homogeneous velocity transformation along an axis by means of a suitable space–time translation, together with suitable rotations of the space-axes of the two observers.

II. THE RECIPROCITY RELATION

As indicated in the introduction, we start from the following assumptions:

(i) *the principle of relativity, which states the equivalence of all inertial systems as regards the formulation of the laws of nature;*
(ii) *the homogeneity of space–time;*
(iii) *the isotropy of space.*

We denote by x the position at which an event takes place and by t the time at which it happens, as viewed by an inertial observer S, and by x', t' the corresponding space–time coordinates of the same event, as viewed by another inertial observer S'.

The homogeneity assumption comes into our considerations in that it implies that the transformation equations which furnish x' and t' as functions of x and t are linear.[7]

In order to prove this assertion, let us employ the notation ξ for the two-vector (x, t) and write the transformation which connects S to S' as

$$\xi' = f(\xi). \tag{1}$$

Since we have decided to restrict our considerations to transformations which conserve the space–time origin, we should require $f(0) = 0$. Here, however, this condition is dropped for the sake of generality.

The homogeneity of space–time requires that a space–time translation T not affect the relation between the two observers and thus leaves Eq. (1) invariant. Denoting by T_α and $T_{\alpha'}$ the representations of T relative to S and S', respectively, we express this property by the relation

$$f(T_\alpha \xi) = T_{\alpha'} f(\xi) \tag{2}$$

or

$$f(\xi + \alpha) = f(\xi) + \alpha', \tag{3}$$

where $\alpha = (\alpha_x, \alpha_t)$, $\alpha' = (\alpha_{x'}, \alpha_{t'})$, and α' depends on f and α but not on ξ.

Taking $\xi = 0$ in (3), we get

$$f(\alpha) = f(0) + \alpha'. \tag{4}$$

Substituting (4) into (3), we obtain

$$f(\xi + \alpha) = f(\xi) + f(\alpha) - f(0). \tag{5}$$

Subtracting $f(0)$ from both sides and setting $g(\xi) = f(\xi) - f(0)$, we have

$$g(\xi + \alpha) = g(\xi) + g(\alpha), \tag{6}$$

for arbitrary ξ and α. From this equation, provided we only assume that g is continuous at the origin, we get that

$$g(k\xi) = kg(\xi), \tag{7}$$

where k is a real number. The proof is quite standard and is given in Appendix A.

Relations (6) and (7) state that g is linear and homogeneous.

We thus write the relation between the pair (x, t) and the pair (x', t') in the form

$$x' = a(v)x + b(v)t,$$
$$t' = c(v)x + d(v)t, \tag{8}$$

where v denotes the velocity of the frame S' with respect to the frame S. This velocity, which we call the *direct velocity* for the pair (S, S'), is given by

$$v = -b(v)/a(v). \tag{9}$$

For the sake of simplicity, we confine ourselves in the rest of this section to the consideration of the case when the space axes of the two observers have the same orientation and their times flow in the same direction,

[6] See, for example, Refs. 2(q, s, aa).

[7] The question of the linearity of the transformation formulas has long been debated in the literature [see, for example, Refs. 1, and 2(d–g, k, m, r, s, y, aa)]. If one does not impose from the outset the existence of an invariant velocity, then the principle of inertia, which implies that a motion which appears uniform to an inertial observer ($x = a_1 t + a_2$) must appear uniform to any other inertial observer ($x' = a_1' t' + a_2'$), is not sufficient to ensure that the transformations are linear. To obtain this result, an additional assumption is needed; namely, that an event of finite space–time coordinates is transformed into an event of finite space–time coordinates. Further, one has to require that the transformation functions be differentiable up to the third order (compare Ref. 2r). It was Einstein (Ref. 1) who first justified the linearity property by an appeal to space–time homogeneity. The argument, however, is rather obscure. Here we give a simple proof of linearity which utilizes a formulation of the homogeneity principle first given by Lalan in Ref. 2k and which is particularly appealing.

which implies the relations[8]

$$\partial x'/\partial x = a(v) > 0,$$
$$\partial t'/\partial t = d(v) > 0, \tag{10a}$$
$$\partial x/\partial x' = d(v)/[a(v)(d(v) + vc(v))] > 0,$$
$$\partial t'/\partial t = 1/(d(v) + vc(v)) > 0. \tag{10b}$$

Then, supposing that both observers use the same unit of time and the same unit of length,[9] the coefficients of (8) are uniquely determined functions of v which, by the principle of relativity, *do not depend on S*.

Denoting by w the *reciprocal velocity*, namely, the velocity of S with respect to S', we have

$$w = b(v)/d(v) = \varphi(v), \tag{11}$$

and our purpose then is to show that the principles of relativity and of isotropy of space, together with some continuity assumptions to be specified later, are sufficient by themselves to arrive at the conclusion that

$$\varphi(v) = -v. \tag{12}$$

The principle of relativity implies that the set Γ of the allowed velocities of S' relative to S does not depend on S and that the reciprocal velocity is the same function of the direct velocity for all pairs of inertial systems. Hence, together with (11), we can write

$$v = \varphi(w) \tag{13}$$

or

$$\varphi(\varphi(v)) = v. \tag{14}$$

Since $w \in \Gamma$, it is clear from (13) that the range of the function φ is equal to its domain Γ. Then φ is a one-to-one mapping of Γ onto Γ. Indeed, if $\varphi(a) = \varphi(b)$, we get from (14) that $a = \varphi(\varphi(a)) = \varphi(\varphi(b)) = b$.

Contrary to a widely held opinion,[3] Eq. (14) is the *only* condition imposed by the principle of relativity on the function φ. This condition is already strongly restrictive on the possible forms of φ, but it by no means alone implies relation (12). For example, the equation

$$w = \varphi(v) = -v/[1 - (v/c)], \tag{15}$$

which is pertinent to one of the cynematics which are compatible with the principle of relativity,[2k] satisfies relation (14) without having the form (12).

[8] Note that conditions (10b) are *a priori* independent of (10a).

[9] It is easy to devise conceptual experiments by which the standards of length and time of the two observers can be made the same. For example, we can make sure that S and S' use the same time standard, if both observers assume as unit of time the mean life of a given unstable particle measured at rest in the laboratory of each of the two observers.

It is precisely with the hope of eliminating solutions of this kind that we resort to the principle of isotropy of space. The main result of applying this principle is the proof that φ is an odd function of v and we see that this property, together with (14) and a physically reasonable assumption concerning the domain of φ and its continuity properties, is sufficient to obtain the result that the reciprocal velocity is given by (12).

In our case, space is one-dimensional and its isotropy means that no one orientation along the x axis should be considered in preference to the other. This assertion is now made precise by stating the isotropy principle in two equivalent forms. The first has a more formal character and concerns the effect that the inversion of the space axes has on the set of transformations (8). The second one, which might be physically more appealing, is based on simple conceptual experiments of a type frequently employed in discussions of the theory of relativity.

We state the isotropy principle in the first form by asserting that if two frames S and S' are connected by a transformation (8), then the two frames $\bar{S}$ and $\bar{S}'$ obtained from the preceding ones by inverting the direction of the x axis are connected by a transformation of the same type. Therefore,

$$\bar{x}' = a(\bar{v})\bar{x} + b(\bar{v})\bar{t},$$
$$\bar{t}' = c(\bar{v})\bar{x} + d(\bar{v})\bar{t}, \tag{16}$$

where $\bar{v}$ is the velocity of $\bar{S}'$ relative to $\bar{S}$. On the other hand, $\bar{x}' = -x'$, $\bar{t}' = t'$, $\bar{x} = -x$, $\bar{t} = t$, so that

$$\bar{x}' = a(v)\bar{x} - b(v)\bar{t},$$
$$\bar{t}' = -c(v)\bar{x} + d(v)\bar{t}, \tag{17}$$

from which we conclude that $\bar{v} = b(v)/a(v) = -v$. Hence, Γ is symmetric and, by comparison with (17),

$$\begin{aligned} a(-v) &= a(v),\\ b(-v) &= -b(v),\\ c(-v) &= -c(v),\\ d(-v) &= d(v). \end{aligned} \tag{18}$$

Then, by (11),

$$\varphi(-v) = -\varphi(v), \tag{19}$$

i.e., φ is an odd function of v.

Consider now the following conceptual experiments:

1. Let T be a rod at rest in S', the end points of which occupy the positions x_1' and x_2'. S measures the length of T by marking the positions x_1 and x_2 that the end points of the rod occupy at a given time t. From the first of the equations in (8) and, again, in (10a) we see that the ratio between the length l' of the rod at

rest (as measured by S') and the length l of the rod in motion (as measured by S) is given by

$$l'/l = a(v). \tag{20}$$

2. Next, let Φ be a phenomenon which takes place at the point x and lasts from time t_1 to time t_2 as observed by S (e.g., we may think of the life of an unstable particle produced at rest at x at time t_1 and decaying at time t_2). By the second of the equations in (10a), the same phenomenon, as observed by S', starts at x_1' at time t_1' and ends at x_2' at time t_2', where (x_1', t_1') and (x_2', t_2') are the transformed coordinates of (x, t_1) and (x, t_2), respectively (in our example, the particle is produced in flight at x_1' at time t_1' and then moves to point x_2', where it decays at time t_2'). From the second of the equations in (8) we see that the ratio between the durations D' and D of Φ (lifetimes of the particle) as measured by S' and S, respectively, is given by

$$D'/D = d(v). \tag{21}$$

3. Finally, let $\tilde{\Phi}$ be another phenomenon which takes place at the point x' and lasts from time t_1' to time t_2' as observed by S', and let (x_1, t_1) and (x_2, t_2) be the transformed coordinates of (x', t_1') and (x', t_2'), respectively. The duration of Φ as measured by S can be determined by means of the following equations:

$$\begin{aligned} a(v)x_1 + b(v)t_1 &= a(v)x_2 + b(v)t_2, \\ t_1' &= c(v)x_1 + d(v)t_1, \\ t_2' &= c(v)x_2 + d(v)t_2, \end{aligned} \tag{22}$$

and, by (9) and by the second of the equations in (10b), the ratio of the durations $\tilde{D}'$ and $\tilde{D}$ of $\tilde{\Phi}$ as measured by the two observers is readily seen to be

$$\tilde{D}'/\tilde{D} = d(v) + vc(v). \tag{23}$$

We state the second version of the principle of isotropy of space by assuming that if v is an allowed velocity, $-v$ is allowed as well (hence the symmetry of Γ), and by requiring that the ratios (20), (21), and (23) are independent of the direction of the motion of S' relative to S, provided that the magnitude of the velocity remains the same, and thus are left unaltered when v is changed to $-v$.[10] This condition implies

$$\begin{aligned} a(-v) &= a(v), \\ d(-v) &= d(v), \\ d(-v) - vc(-v) &= d(v) + vc(v). \end{aligned} \tag{24}$$

Taking into account (9), we see that relations (24) are equivalent to relations (18), and then (19) follows.

[10] The same requirement, as regards only the ratio (20), was originally imposed by Frank and Rothe in their derivation of the Lorentz transformations (cf. Ref. 2a).

We make two further assumptions before we derive the reciprocity relation. These are:

(a) The domain Γ of the function φ is an interval on the real line;

(b) φ is continuous on Γ.

In other words, it is assumed that if v_1 and v_2 are two allowed velocities, any velocity v which is comprised between v_1 and v_2 is again allowed, and that the reciprocal velocity is a continuous function of the direct velocity. The physical plausibility of these two conditions is obvious.

Since φ is a continuous one-to-one mapping of Γ onto itself and Γ is connected, then, from a well-known theorem of analysis, we can state that φ is either a strictly increasing or a strictly decreasing function of v.[11]

Suppose first that φ is strictly increasing. Let $v \in \Gamma$; then $w = \varphi(v) \in \Gamma$. Assume that $v < w$; then $\varphi(v) < \varphi(w)$ and, by (14), $w < v$, which is absurd. We can conclude in the same way $v > w$, so that

$$\varphi(v) = v. \tag{25}$$

If φ is supposed to be strictly decreasing, set $\psi = -\varphi$. Then ψ is strictly increasing and, by (19) and (14), satisfies $\psi(\psi(v)) = v$. Applying to ψ the same argument as before, we obtain $\psi(v) = v$, i.e., Eq. (12).

The choice of Eq. (25) leads to the transformation formulas

$$\begin{aligned} x' &= a(v)x - va(v)t, \\ t' &= c(v)x - a(v)t, \end{aligned} \tag{26}$$

while the choice of Eq. (12) leads to

$$\begin{aligned} x' &= a(v)x - va(v)t, \\ t' &= c(v)x + a(v)t. \end{aligned} \tag{27}$$

Formulas (26) are incompatible with (10). Hence, for two observers whose space axes have the same orientation and whose times flow in the same direction, (12) must necessarily hold and the transformation formulas are given by (27).

Our task of proving the reciprocity relation has thus been completed.

III. EXPLICIT FORM OF THE TRANSFORMATION EQUATIONS

Formulas (27) contain the two as yet undetermined functions $a(v)$ and $c(v)$. However, it is seen at once that $c(v)$ can be expressed in terms of v and $a(v)$.

[11] See, for example, J. Dieudonné, *Foundations of Modern Analysis* (Academic Press Inc., New York, 1960), Theorem 4.2.2. For the reader's convenience the proof is given with some detail in Appendix B.

Indeed, consider the inverse transformations

$$x = \Delta^{-1}(v)a(v)x' + \Delta^{-1}(v)va(v)t',$$
$$t = -\Delta^{-1}(v)c(v)x' + \Delta^{-1}(v)a(v)t', \qquad (28)$$

where

$$\Delta(v) = a(v)\{a(v) + vc(v)\}. \qquad (29)$$

By the reciprocity relation, (28) can also be written in the form

$$x = a(-v)x' + va(-v)t',$$
$$t = c(-v)x' + a(-v)t', \qquad (30)$$

whereby, using (18),

$$a(v) = 1/(a(v) + vc(v)),$$

so that

$$c(v) = (1/v)\{a^{-1}(v) - a(v)\}.$$

Then the transformations (27) read

$$x' = a(v)x - va(v)t,$$
$$t' = (1/v)\{a^{-1}(v) - a(v)\}x + a(v)t. \qquad (31)$$

To interpret (26) we proceed as above, by using (25) instead of (12), and obtain

$$x' = a(v)t - va(v)t,$$
$$t' = -[(1/v)\{a^{-1}(v) - a(v)\}x + a(v)t]. \qquad (32)$$

Hence, (26) is obtained from (27) by an inversion of the time of S', and this explains why (26) corresponds to the choice $\varphi(v) = v$.

The transformation formulas, which connect S to an observer obtained from S' by inverting the orientation of the space-axis, are

$$x' = -[a(v)x - va(v)t],$$
$$t' = (1/v)\{a^{-1}(v) - a(v)\}x + a(v)t, \qquad (33)$$

whereas if S' is subjected to both a space and a time inversion, then

$$x' = -[a(v)x - va(v)t],$$
$$t' = -[(1/v)\{a^{-1}(v) - a(v)\}x + a(v)t]. \qquad (34)$$

It is the essence of the principle of relativity that the set of *all* transformations (31)–(34), as v varies in Γ, forms a group $\mathfrak{L}$. From this property one can derive the explicit form of $a(v)$. In fact, let us compose two transformations of type (31), such that

$$\begin{pmatrix} a(v) & -va(v) \\ (1/v)(a^{-1}(v) - a(v)) & a(v) \end{pmatrix}\begin{pmatrix} a(v') & -v'a(v') \\ (1/v')(a^{-1}(v') - a(v')) & a(v') \end{pmatrix}$$
$$= \begin{pmatrix} a(v)a(v') - (v/v')a(v)(a^{-1}(v') - a(v')) & -(v + v')a(v)a(v') \\ (a(v')/v)(a^{-1}(v) - a(v)) + (a(v)/v')(a^{-1}(v') - a(v')) & a(v)a(v') - (v'/v)a(v')(a^{-1}(v) - a(v)) \end{pmatrix}$$
$$= A(v, v'). \qquad (35)$$

The resulting transformation must be of one of the four types (31)–(34) for some relative velocity v''. However, since the determinant of transformations (31) and (34) is $+1$, whereas the determinant of transformations (32) and (33) is -1, then types (32) and (33) must be ruled out.

Since the diagonal elements of the matrices of both transformations (31) and (34) are equal, we must have

$$a(v)a(v') - (v/v')a(v)(a^{-1}(v') - a(v'))$$
$$= a(v)a(v') - (v'/v)a(v')(a^{-1}(v) - a(v)),$$

i.e.,

$$(1/v^2)\{1 - a^{-2}(v)\} = (1/v'^2)\{1 - a^{-2}(v')\}, \quad (36)$$

whence

$$(1/v^2)\{1 - a^{-2}(v)\} = K, \qquad (37)$$

where K is a universal constant having the dimensions of an inverse-square velocity. Then, since $a(v)$ is positive,

$$a(v) = 1/(1 - Kv^2)^{\frac{1}{2}}. \qquad (38)$$

The composite velocity v'' is the negative ratio between the second and the first element of the matrix $A(v, v')$:

$$v'' = (v + v')/[1 - (v/v')\{a^{-2}(v') - 1\}]$$
$$= (v + v')/(1 + Kvv'). \qquad (39)$$

Three cases are to be considered:

(A) $K > 0$. Set $c = (K)^{-\frac{1}{2}}$ and formulas (31) become

$$x' = [1 - (v^2/c^2)]^{-\frac{1}{2}}x - \{v[1 - (v^2/c^2)]^{-\frac{1}{2}}\}t,$$
$$t' = -\{(v/c^2)[1 - (v^2/c^2)]^{-\frac{1}{2}}\}x + [1 - (v^2/c^2)]^{-\frac{1}{2}}t, \qquad (40)$$

and v varies in the domain $\Gamma = (-c, c)$. Equations (40) are the ordinary proper orthochronous Lorentz transformations.

(B) $K = 0$. Formulas (31) become

$$x' = x - vt,$$
$$t' = t, \qquad (41)$$

and v varies in the domain $\Gamma = (-\infty, +\infty)$. These are the Galilean transformations.

(C) $K < 0$. Set $c = (-K)^{-\frac{1}{2}}$ and formulas (31) become

$$x' = [1 + (v^2/c^2)]^{-\frac{1}{2}}x - \{v[1 + (v^2/c^2)]^{-\frac{1}{2}}\}t,$$
$$t' = \{(v/c^2)[1 + (v^2/c^2)]^{-\frac{1}{2}}\}x + [1 + (v^2/c^2)]^{-\frac{1}{2}}t, \qquad (42)$$

and v varies in the domain $\Gamma = (-\infty, +\infty)$.

Set $x^1 = x$, $x^0 = ct$, and $tg\alpha = v/c$ $(-\pi/2 < \alpha < \pi/2)$, and (42) becomes

$$x'^1 = (\cos\alpha)x^1 - (\sin\alpha)x^0,$$
$$x'^0 = (\sin\alpha)x^1 + (\cos\alpha)x^0. \tag{43}$$

Hence, in contrast to the Lorentz transformations (40), which, as is well known, are hyperbolic rotations in the plane (x, t), transformations (42) are ordinary circular rotations. Since α is confined to the interval $(-\pi/2, \pi/2)$, it is clear that *they do not form a group*. If we let α vary from $-\pi/2$ to $3\pi/2$, so as to obtain the full group, we can easily see that we are led to introduce also the transformations

$$x' = -([1 + (v^2/c^2)]^{-\frac{1}{2}}x - \{v[1 + (v^2/c^2)]^{-\frac{1}{2}}\}t),$$
$$t' = -(\{(v/c^2)[1 + (v^2/c^2)]^{-\frac{1}{2}}\}x + [1 + (v^2/c^2)]^{-\frac{1}{2}}t), \tag{44}$$

which are obtained from (42) by inverting both the space and the time axis of S'.

The rotation group (43) translates into mathematical form a complete isotropy of space–time, so that the two directions in time are completely equivalent as well as the two directions in space. On the other hand, if one believes that *there is an intrinsic arrow in the direction of flow of time*, so that time reversal is regarded as a purely mathematical operation which cannot be physically realized, one obtains a strong argument to rule out the transformations (42). Close to this argument is the one set forth by Lalan,[2k] who postulates that if two events take place at the same point in space with respect to a given observer, their time order must be the same for all observers. Alternatively, we could postulate that the relation which states that the space axes have the same orientations and that the times flow in the same direction is transitive, which amounts to assuming that the set of proper orthochronous transformations (31) is by itself a group.

Two other curious features of transformations (43) can be read out in the formula of composition of velocities (39) which, in the present case, has the form

$$v'' = (v + v')/[1 - (vv'/c^2)]. \tag{45}$$

First, by composing two finite velocities v and v' such that $vv' = c^2$, one obtains an infinite velocity v''. Second, by composing two positive velocities v and v' such that $vv' > c^2$, one obtains a negative velocity v''. Some authors[2s,w] use these properties as an argument to exclude transformations (42). In our opinion, however, an argument of this kind is not so convincing as the preceding ones in such a general context, because there are not sufficient reasons of principle to exclude the appearance of phenomena such as those described above. Besides, it is to be noted that peculiarities of this type also appear in the Lorentz case, for which (39) reads

$$v'' = (v + v')/[1 + (vv'/c^2)]. \tag{46}$$

Indeed, if, following some recent ideas,[12] one conjectures the existence of faster-than-light particles (tachyons) and interprets (46) as the transformation formula for the tachyon velocity (v' = particle velocity as measured by S'; v'' = particle velocity as measured by S), it is easily seen that, fixing v very small and negative, we can transform a very large, greater than c, and positive v' into a very large, greater than c, and negative v''. Further, there always exists a reference frame relative to which a tachyon propagates instantaneously. These features are just as curious as those which have been discussed above in connection with formula (45). Notwithstanding, this has not prevented some authors from considering the possibility that faster-than-light particles really exist, on the grounds that the usual objections to such particles are ultimately found to be unconvincing when subjected to critical analysis.

Once we agree to reject formulas (42), we are left with the problem of the choice between the Lorentz transformations (40) and the Galilean transformations (41). As is well known, the Lorentz transformations admit one and only one invariant velocity which is equal to c. In the limit when this velocity is taken to be infinite, one obtains the Galilean transformations. Hence the above problem of choice can be solved only by experience and involves the search for an invariant velocity in nature. The experimental evidence for the existence of signals which travel with a *finite invariant velocity* (such as the electromagnetic waves in vacuo) leads us to rule out the Galilean transformations in favor of the Lorentz ones. In these, of course, the numerical value to be assigned to c is the experimentally measured value of this invariant velocity, namely, the value of the velocity of propagation of electromagnetic disturbances in empty space.

Formally, the rotation transformations (42) correspond instead to the appearance of an invariant imaginary velocity c. This is expressed by the property that they are the linear transformations which conserve the positive-definite quadratic form $x^2 + c^2t^2$, while the Lorentz transformations are those which conserve the indefinite form $x^2 - c^2t^2$. In a

[12] O. M. P. Bilaniuk, V. K. Deshpande, and E. C. G. Sudarshan, Am. J. Phys. **30**, 718 (1962); S. Tanaka, Progr. Theoret. Phys. (Kyoto) **24**, 171 (1960); G. Feinberg, Phys. Rev. **159**, 1089 (1967), and unpublished; R. Newton, Phys. Rev. **162**, 1274 (1967); M. E. Arons and E. C. G. Sudarshan, *ibid.* **173**, 1622 (1968).

four-dimensional space–time the corresponding conserved forms are $x^2 + y^2 + z^2 + c^2t^2$ and $x^2 + y^2 + z^2 - c^2t^2$ and the appropriate groups are the orthogonal group in four dimensions $O(4)$ and the Lorentz group $O(3, 1)$. The characteristic of $O(4)$, that a transformation containing both space reflection and time inversion can be joined continuously to the identity, corresponds to the topological property that, while $O(3, 1)$ has four connected components, $O(4)$ has only two.

IV. CONCLUSION

By making use of the principle of relativity and of the isotropy of space, we have deduced in a simple but rigorous way the reciprocity relation for the relative motion of two inertial reference frames, which is usually assumed as a postulate in the standard derivations of the Lorentz transformations without the principle of invariance of light velocity. For completeness we have then given the usual deduction of the transformation equations by using their group property. We have put forward some alternative arguments to rule out the transformations with invariant imaginary velocity. From a logical viewpoint these arguments might seem more appealing than those previously given by other authors.

APPENDIX A

Let g be a mapping of R^n into itself such that

$$g(\xi + \zeta) = g(\xi) + g(\zeta). \tag{A1}$$

If n is a positive integer, we get by induction, from (A1), that

$$g(n\xi) = ng(\xi). \tag{A2}$$

As $g(0) = 0$, $g(-\xi) = -g(\xi)$, so that (A2) holds equally well for n any integer.

Next, for any rational $r = m/n$, set $m\xi = n\eta$. Then

$$mg(\xi) = g(m\xi) = g(n\eta) = ng(\eta)$$

and thus

$$g(r\xi) = rg(\xi). \tag{A3}$$

Assume now that g is continuous at the origin. This property, together with (A1), implies that g is continuous everywhere. Then, let k be any real number and $\{k_n\}$ be a sequence of rationals which converges to k. So

$$k_n\xi \xrightarrow[n\to\infty]{} k\xi$$

and, by continuity,

$$g(k_n\xi) \xrightarrow[n\to\infty]{} g(k\xi).$$

But

$$g(k_n\xi) = k_n g(\xi) \xrightarrow[n\to\infty]{} kg(\xi),$$

so that

$$g(k\xi) = kg(\xi). \tag{A4}$$

(A1) and (A4) state that g is an endomorphism of the vector space $\mathbf{R}^n$.

APPENDIX B

We recall the following two results of general topology [cf. Ref. (11), Theorems 3.19.7 and 3.19.1].

Proposition 1: The continuous image of a connected topological space is connected.

Proposition 2: A necessary and sufficient condition for a subset A of the real line to be connected is that A is an interval.

In the following, if s and t are any two real numbers, $[s, t]$ will denote the closed interval $\{x : s \leq x \leq t\}$, if $s \leq t$, and the closed interval $\{x : t \leq x \leq s\}$, if $t \leq s$.

In order to prove that the mapping φ is strictly monotone, consider two fixed points p and q of Γ such that $p < q$. Since φ is one to one, we can exclude $\varphi(p) = \varphi(q)$ and suppose, for instance, that $\varphi(p) < \varphi(q)$. Let r be any other point of Γ, $r \neq p$, $r \neq q$. We prove

$$r < p \Rightarrow \varphi(r) < \varphi(p), \tag{B1a}$$

$$p < r \Rightarrow \varphi(p) < \varphi(r), \tag{B1b}$$

and

$$r < q \Rightarrow \varphi(r) < \varphi(q), \tag{B2a}$$

$$q < r \Rightarrow \varphi(q) < \varphi(r). \tag{B2b}$$

Indeed, let, for example, $p < r$. We have $\varphi(r) \neq \varphi(p)$, as implied by φ being one to one, and suppose it to be $\varphi(r) < \varphi(p)$. Since φ is continuous, by propositions 1 and 2, $\varphi([r, q])$ is an interval, so that $\varphi([r, q]) \supseteq [\varphi(r), \varphi(q)]$, whereby $\varphi(p) \in \varphi([r, q])$ because $\varphi(r) < \varphi(p) < \varphi(q)$. Then there is a $p' \in [r, q]$ such that $\varphi(p') = \varphi(p)$, and this is incompatible with φ being one to one because $p' \neq p$, as implied by $p < r$, $p < q$. (B1b) is thus proved.

(B1a), (B2a), and (B2b) are proved in a similar way.

Let now y and y' be any two points of Γ with $y < y'$. Choose s such that $y < s < y'$. Three cases are possible: $s = p$, $s < p$, and $p < s$. In the first case, apply (B1a) and (B1b) to get $\varphi(y) < \varphi(y')$. In the second case, (B1a) implies $\varphi(s) < \varphi(p)$ and we can again apply (B1a) and (B1b) with s in place of p to obtain $\varphi(y) < \varphi(y')$. In the last case, (B1b) gives $\varphi(p) < \varphi(s)$, and use of (B2a) and (B2b) with s in place of q gives again $\varphi(y) < \varphi(y')$. Hence $y < y' \Rightarrow \varphi(y) < \varphi(y')$, and φ is strictly increasing. One can show in the same way that the alternative $\varphi(q) < \varphi(p)$ implies that φ is strictly decreasing.