Relativity 1

Disclaimer: These lecture notes are not meant to replace the course textbook. The content may be incomplete. Some topics may be unclear. These notes are only meant to be a study aid and a supplement to your own notes. Please report any inaccuracies to the professor.

Newtonian Relativity

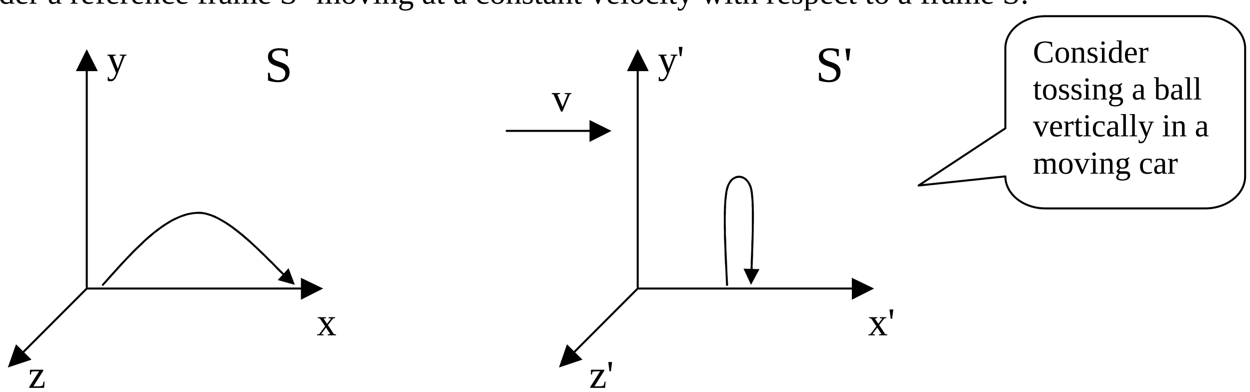
Galileo and Newton described the motion of objects with respect to a particular **reference frame**, which is basically a coordinate system attached to a particular observer.

A **reference frame** in which Newton's Laws hold is called an **inertial frame**. It is a frame that is not accelerating.

Newtonian Principle of Relativity (Galilean Invariance):
If Newton's Laws hold in one inertial frame, they also hold in a reference frame moving at a constant velocity relative to the first frame. So the other frame is also an inertial frame. We can see this if we make a Galilean transformation:

Galilean Transformation

Consider a reference frame S' moving at a constant velocity with respect to a frame S:



$$x = x' + vt' \qquad x' = x - vt$$
$$y = y' \qquad y' = y$$
$$z = z' \qquad z' = z$$
$$t = t' \qquad t' = t$$

These transformation equations show you how to convert a coordinate measured in one reference frame to the equivalent coordinate in the other reference frame. Implicit in a Galilean transformation is that time is universal (time runs at the same rate in all frames).

Now consider the action of a force in one reference frame. For example, the force of gravity causes a dropped ball to accelerate:

y component:

$$F_y{}' = ma_y{}' = m\frac{d^2y'}{dt'^2}$$

But since $y' = y$ (and $t' = t$)

$a_y{}' = a_y$ and $F_y{}' = F_y$

x component:

$$F_x{}' = ma_x{}' = m\frac{d^2x'}{dt'^2} = m\frac{d^2}{dt^2}(x - vt) = m\frac{d^2x}{dt^2} = F_x$$

$a_x{}' = a_x$ and $F_x{}' = F_x$

Since the acceleration of the ball is the same in each reference frame, and thus the force acting on the ball, Newton's Laws are valid in both frames. Each is an inertial frame.

Note that since the force is identical in each frame, there is no way to detect which frame is moving and which is not. You can only detect relative motion. For example, if a jet flies west at 1000 mph at the equator, is the jet moving or is the Earth moving? The jet flies over the surface of the Earth, but with respect to the Sun the jet is not moving and the Earth is turning beneath it! The fact that we cannot detect absolute motion is known as ***Relativity***. It is only relative motion that matters.

Example: Consider tossing a ball forward from a moving car at a velocity *v'* with respect to the reference frame of the car. What is the velocity of the ball with respect to the sidewalk along the road?

We need to know how to transform velocities. If we assume that the car is moving along the *x* axis at a velocity *v* with respect to the reference frame of the road, then

$$x = x' + vt$$

according to a Galilean Transformation. If we differentiate this with respect to time, we get:

$$\frac{dx}{dt} = \frac{dx'}{dt} + v$$

So, the velocity of the ball is *v'*+*v*. It is the sum of the car's velocity and the velocity of the ball with respect to the car. This should agree with our common sense.

Now suppose that instead of a ball we throw a light beam forward from the car. Light is an electromagnetic wave, and according to Maxwell's Equations it travels at a velocity $c = 3.0 \times 10^8\ \mathrm{m/s}$ in vacuum. For example, you could derive the following equation:

$$\frac{\partial^2 E_x}{\partial x^2} = \mu_0 \varepsilon_0 \frac{\partial^2 E_x}{\partial t^2}$$

$$\Rightarrow E_x = E_0 \sin[k(x - vt)] \quad \text{where} \quad v = c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$$

This is the velocity with respect to the car, but what about the velocity with respect to the sidewalk? Would it be $c+v$? That would agree with our common sense, but not with Maxwell's Equations. Maxwell's Equations state that the speed of light is c and only c, but for which reference frame does it refer to?

For clarification of the issue, let's consider **sound waves**, which are another form of a traveling wave. Sound waves are pressure waves. Pressure is a measure of how hard molecules push on a wall (force per unit area), so obviously you need some molecules around to have pressure, and thus pressure waves. Therefore, sound waves require a medium to propagate. The speed of sound at normal temperature and pressure in air is 343 m/s, or about 765 mph.

In our car example, we could consider honking the horn instead of turning on the headlights. We then create sound waves that propagate forward from the vehicle. The speed of sound is 343 m/s as we have noted, but this is the speed with respect to the propagation medium. If the air is still with respect to the sidewalk, then the speed of the sound wave is 343 m/s with respect to the sidewalk and **not** the horn on the car. In fact, if our car had a rocket strapped on, it could exceed the speed of sound and overtake its own emitted sound wave. (You then create a shock wave, which gives rise to a sonic boom. By the way, this car experiment was actually done recently!)

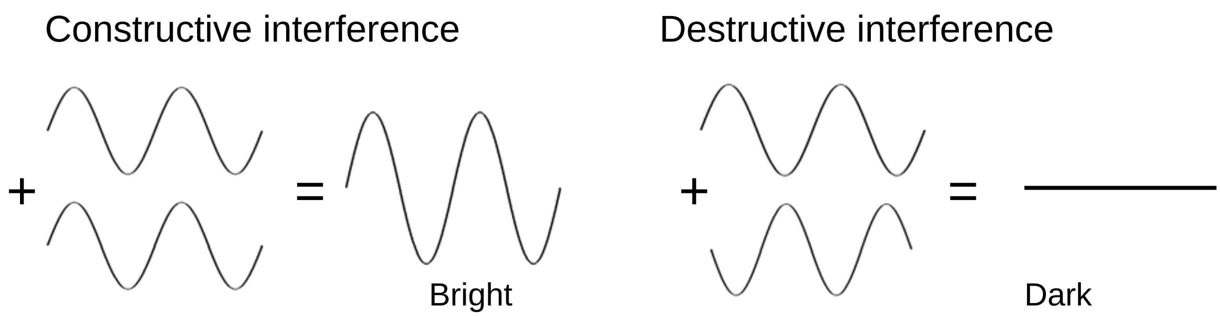
So honking the horn is not the same as tossing a ball forward. The velocity of the sound waves is always 343 m/s with respect to the reference frame of the air. It did not obey a Galilean Transformation because it picks out a special reference frame.

Now let's return to the light from the car's headlights. If the light beam acts like a ball thrown forward, we would measure a different velocity for the light depending on whether we were in the car or on the sidewalk. Also, Maxwell's Equations would have to be modified to account for a velocity different than c for all reference frames other than the one it apparently describes (does it apply to the car reference frame or the sidewalk?) If the light waves act like sound waves, then what is the propagation medium? Is it the air on the Earth? It can't be because light can propagate in a vacuum. So instead, it was proposed in the 19th century that light propagates through **ether**—some sort of medium—although nobody knew what this ether was. It was supposed that this ether might be at rest with respect to the solar system, or maybe the galaxy. In any case, the Earth would move through this ether, and we should observe light traveling at a speed different than c.

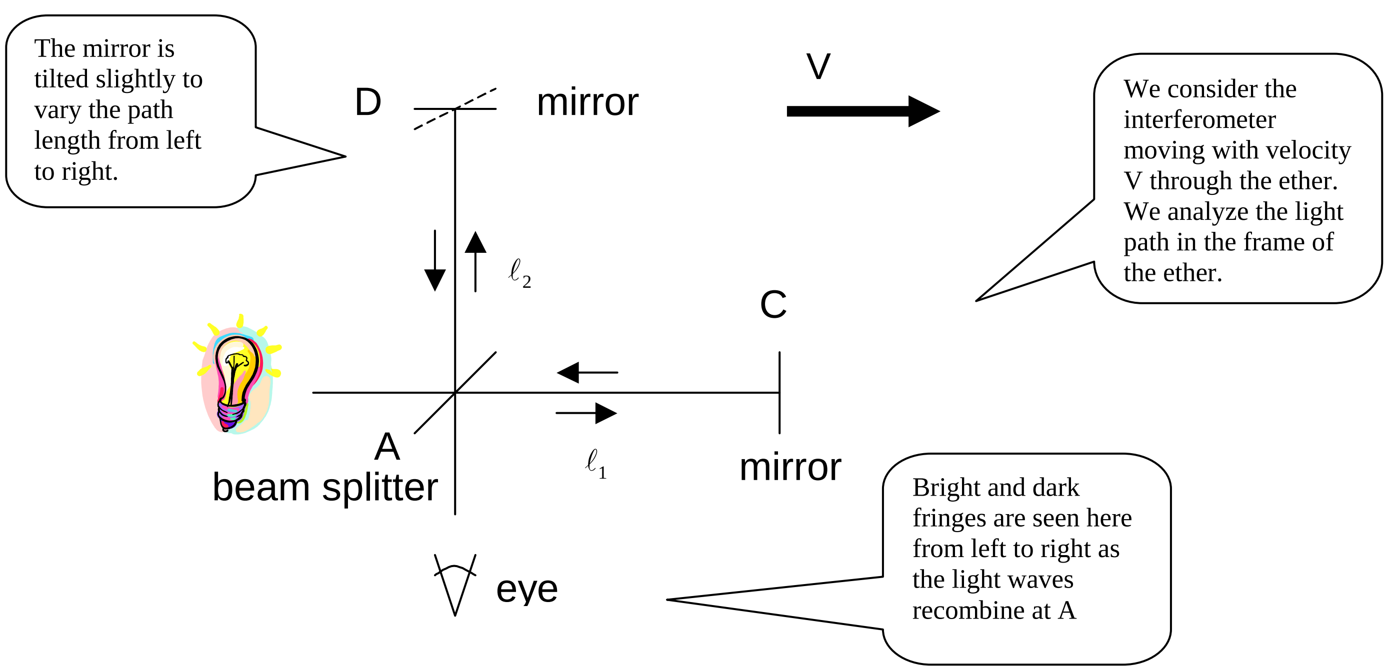
The issue was settled experimentally, but it took Einstein to put everything in the right perspective. The answer will surprise you. Light waves do act like balls thrown forward, but the assumptions behind the Galilean Transformation are wrong. Not only that, there is nothing wrong with Maxwell's Equations. The speed of light is always c no matter who measures it. But I'm getting ahead of myself…

The Michelson-Morely Experiment

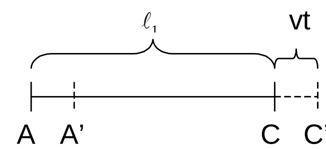
The question of whether ether exists was settled by the Michelson-Morely experiment(s) in 1887. An **interferometer** was used to separate a light beam into two paths of possibly different length and then recombined. Since light is a wave, it exhibits the phenomenon of interference when multiple waves are combined. If two light waves are completely in phase, then the amplitude of each wave adds **constructively**. If they are completely out of phase, the amplitudes subtract **destructively**. Interferometers use monochromatic light so that the light wave consists of nearly a single wavelength. (Today we would use a laser).



The Michelson-Morely interferometer has two paths at right angles with respect to each other. It is at rest in a laboratory, presumably traveling through the ether. Considering that the velocity of light is *c* with respect to the ether, the distance light travels along each path is different even if the length of each "arm" is the same. Let's calculate the time it takes light to travel the length of each arm (and back) assuming that one arm is aligned with the velocity vector through the ether.



Calculate the time it takes to travel the horizontal path from $\mathrm{A} \leftrightarrow \mathrm{C}$:



$$c\,t_{A\to C} = \ell_1 + v\,t_{A\to C} \quad \Rightarrow \quad t_{A\to C} = \frac{\ell_1}{c-v}$$

$$c\,t_{C\to A} = \ell_1 - v\,t_{C\to A} \quad \Rightarrow \quad t_{C\to A} = \frac{\ell_1}{c+v}$$

$$t_1 \equiv t_{A\to C} + t_{C\to A} = \frac{\ell_1}{c-v} + \frac{\ell_1}{c+v} = \frac{2c\ell_1}{c^2-v^2}$$

$$t_1 = \frac{2\ell_1}{c}\frac{1}{1-v^2/c^2}$$

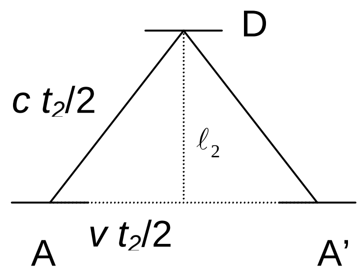
Calculate the time it takes to travel the vertical path from $\mathrm{A} \leftrightarrow \mathrm{D}$ using the Pythagorean theorem:

$$t_2 \equiv t_{A\to D} + t_{D\to A} = 2t_{A\to D} \quad \text{(By symmetry)}$$

$$\ell_2^2 + (vt_2/2)^2 = (ct_2/2)^2$$

$$\ell_2^2 = (c^2 - v^2)t_2^2/4$$

$$t_2 = \frac{2\ell_2}{c}\frac{1}{\sqrt{1-v^2/c^2}}$$



The time difference is:

$$\Delta t = t_2 - t_1 = \frac{2}{c}\left(\frac{\ell_2}{\sqrt{1-v^2/c^2}} - \frac{\ell_1}{1-v^2/c^2}\right)$$

If you multiply this by *c*, then that is the extra distance light must travel along path 2. Even if the length of the arms of the interferometer are the same, there is a time difference because the interferometer is traveling through the ether (and the speed of light is fixed in the ether).

Now consider rotating the entire interferometer by 90 degrees. The horizontal path becomes the vertical path and vice versa. We just swap indices and get:

$$\Delta t' = t_2' - t_1' = \frac{2}{c}\left(\frac{\ell_2}{1-v^2/c^2} - \frac{\ell_1}{\sqrt{1-v^2/c^2}}\right)$$

So the difference in time between path 2 and path 1 changes when we change the orientation of the interferometer with respect to the velocity of the ether. The difference of these differences is:

$$\Delta t' - \Delta t = \frac{2}{c}\left(\frac{\ell_1 + \ell_2}{1 - v^2/c^2} - \frac{\ell_1 + \ell_2}{\sqrt{1 - v^2/c^2}}\right)$$

$$\approx \frac{2(\ell_1 + \ell_2)}{c}\left[\left(1 + v^2/c^2 + \cdots\right) - \left(1 + v^2/2c^2 + \cdots\right)\right]$$

$$\Delta t' - \Delta t \approx \frac{v^2(\ell_1 + \ell_2)}{c^3}$$

Here we have made use of the binomial expansion to approximate the answer:

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots$$

$$(1+x)^n = 1 + nx + \cdots$$

If we multiply this difference of differences by the speed of light, then that corresponds to a shift in the path length difference between arm 1 and arm 2. If we divide that by the wavelength of light, then we have the shift expressed as a fraction of a wavelength:

$$\frac{\Delta\lambda}{\lambda} = \frac{c}{\lambda}(\Delta t' - \Delta t) = \frac{v^2}{c^2}\frac{(\ell_1 + \ell_2)}{\lambda}$$

Note that $c/\lambda = f$, the frequency of the light.

What we expect to see in the Michelson-Morely experiment is a shift in the interference fringes as one rotates the interferometer by 90 degrees. Let's see how big a shift:

$$\ell_1 = \ell_2 = 11 \text{ m}$$

$$v = 3.0 \times 10^4 \text{ m/s} \quad \text{(Earth's orbital velocity)}$$

We assume that the ether is at rest with respect to the solar system

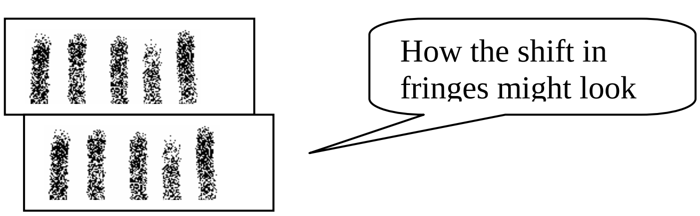
$$c = 3.0 \times 10^8 \text{ m/s}$$

$$f = 5 \times 10^{14} \text{ s}^{-1} \qquad \text{(red light, with } \lambda = 6 \times 10^{-7} \text{ m)}$$

$$\Rightarrow \frac{\Delta\lambda}{\lambda} = 0.4$$

Therefore, we expect to see the fringes move by almost half the distance from one bright fringe to the next. This should be quite noticeable.

Conclusion: No shift was seen! Nor has one been seen ever since 1887. The conclusion must be that the ether does not exist. Light does not require any medium to propagation.



Relativity 2

Disclaimer: These lecture notes are not meant to replace the course textbook. The content may be incomplete or even inaccurate. Some topics may be unclear. These notes are only meant to be a study aid and a supplement to your own notes. Please report any inaccuracies to the professor.

Einstein's Postulates

The absence of any fringe shift in the Michelson-Morely experiment for any orientation of the interferometer and for any time of the year negated the ether hypothesis for light propagation. Light waves are oscillations of the electromagnetic field, and no propagation medium is necessary, unlike sound waves. However, if Galilean transformations are correct, then Maxwell's equations must be modified for every possible reference frame to account for different velocities for the speed of light. Einstein assumed the opposite: that Maxwell's equations are fundamentally correct, but that our intuitive Galilean transformation is not. This led to the following two postulates:

1. The laws of physics, including electromagnetism, are the same in all inertial frames.
2. Every observer measures the same value c for the speed of light (in vacuum) in all inertial frames.

The second postulate is really a consequence of the first, because if Maxwell's equations hold in all inertial frames, then the only possible value for the speed of light is c. These postulates embody Einstein's **Special Theory of Relativity**, first published in 1905 in a paper titled <u>On the Electrodynamics of Moving Bodies</u>. Later he would incorporate gravity and acceleration in his General Theory of Relativity. As in Newtonian Relativity, there is no way to detect absolute motion. Only the relative velocities between two inertial reference frames matters.

These seemingly simple postulates have extraordinary consequences. For example, when you turn on the headlights of a car, the light beam leaves the car at a relative velocity of $c = 3.0 \times 10^8$ m/s. However, someone standing on the sidewalk also measures the speed of the light beam as c independent of the velocity of the car! How can this be? As we shall see, our concepts of space and time must be modified.

Basic Definitions

Events are physical phenomena that occur independent of any reference frame. For example: a flash, explosion, return of a spaceship, or disintegration of a subatomic particle.

Observers record events, both the time and spatial coordinates, in a particular reference frame. For example, Mission Control in Houston marking down the time and location of the splashdown of a space capsule. The reference frame in this case is the Earth.

Simultaneous events occur when the light signals from two events reach an observer at the same time

Relativity of Simultaneity:

Two events simultaneous in one inertial frame are not simultaneous in any other frame. This is a consequence of Einstein's Postulates.

Proper time is the time difference between two events occurring at the same position (Denoted by t_0 or τ).

Rest frame is the inertial frame where two events are only separated by time. The frame in which the proper time is measured

Proper length is the distance between two positions at rest, the length measured in the rest frame. (Denoted by L_0).

Now that we are armed with these definitions, let's explore the consequences of the constancy of the speed of light in all inertial frames.

Time Dilation

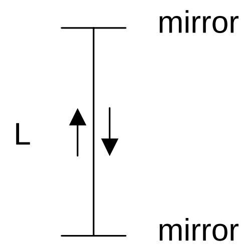
We explore the rate of time in different inertial frames by considering a special kind of clock – a light clock – which is just one arm of an interferometer. Consider a light pulse bouncing vertically between two mirrors. We analyze the time it takes for the light pulse to complete a round trip both in the rest frame of the clock (labeled S'), and in an inertial frame where the clock is observed to move horizontally at a velocity *v* (labeled S).

In the rest frame S'

$$t_1' = \frac{L}{c} = \text{ time up}$$

$$t_2' = \frac{L}{c} = \text{ time down}$$

$$\tau = t_1' + t_2' = \frac{2L}{c}$$



Now put the light clock on a spaceship, but measure the roundtrip time of the light pulse from the Earth frame S:

$$t_1 = \frac{t}{2} = \text{ time up}$$

$$t_2 = \frac{t}{2} = \text{ time down}$$

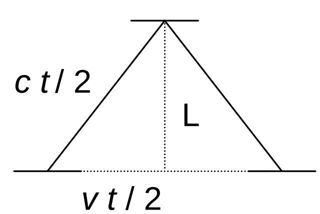
The speed of light is still c in this frame, so

$$L^2 + v^2 t^2/4 = c^2 t^2/4$$

$$L^2 = \left(c^2 - v^2\right) t^2/4$$

$$t^2 = \frac{4L^2}{c^2 - v^2}$$

$$t = \frac{2L}{c}\frac{1}{\sqrt{1-v^2/c^2}} = \frac{\tau}{\sqrt{1-v^2/c^2}}$$



So the time it takes the light pulse to make a roundtrip in the clock when it is moving by us is appears longer than when it is at rest. We say that time is **dilated**. It also doesn't matter which frame is the Earth and which is the clock. Any object that moves by with a significant velocity appears to have a clock running slow. We summarize this effect in the following relation:

$$\boxed{t = \gamma\,\tau \qquad \gamma = \frac{1}{\sqrt{1-v^2/c^2}} \geq 1}$$

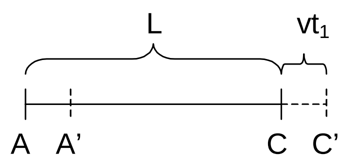
Length Contraction

Now consider using a light clock to measure the length of an interferometer arm. In particular, let's measure the length along the direction of motion.

In the rest frame S'

$$L_0 = \frac{c\tau}{2}$$

Now put the light clock on a spaceship, but measure the roundtrip time of the light pulse from the Earth frame S:



$t_1 =$ time out

$t_2 =$ time back

$t = t_1 + t_2$

$$L + vt_1 = ct_1 \quad \Rightarrow \quad t_1 = \frac{L}{c-v}$$

$$L - vt_2 = ct_2 \quad \Rightarrow \quad t_2 = \frac{L}{c+v}$$

$$t = t_1 + t_2 = \frac{2Lc}{c^2 - v^2} = \frac{2L}{c}\frac{1}{1 - v^2/c^2}$$

$$L = \frac{ct}{2}\left(1 - v^2/c^2\right)$$

But, $t = \dfrac{\tau}{\sqrt{1 - v^2/c^2}}$ from time dilation

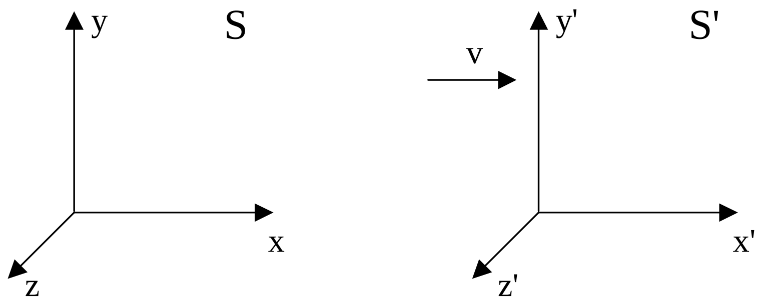
$$\boxed{L = \frac{L_0}{\gamma} \qquad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \geq 1}$$

In other words, the length of the interferometer arm appears contracted when it moves by us. This is known as the **Lorentz-Fitzgerald contraction**. It is closely related to time dilation. In fact, one implies the other, since we used time dilation to derive length contraction.

Time dilation and length contraction are consequences of the assumption that all observers measure the same value for the speed of light. This means that time runs at different rates for different inertial frames. There is no absolute time, time only has a relative meaning. Likewise, length also has only a relative meaning. Everything depends on the relative velocity between two objects. We only notice these strange effects when the velocity is near *c*, however.

The Lorentz Transformation

We are now ready to derive the correct transformation equations between two inertial frames in Special Relativity, which modify the Galilean Transformation. We consider two inertial frames S and S', which have a relative velocity v between them along the x-axis.



Now suppose that there is a single flash at the origin of S and S' at time $t = t' = 0$, when the two inertial frames happen to coincide. The outgoing light wave will be spherical in shape moving outward with a velocity c in *both* S and S' by Einstein's Second Postulate.

$$x^2 + y^2 + z^2 = c^2t^2$$
$$x'^2 + y'^2 + z'^2 = c^2t'^2$$

We expect that the orthogonal coordinates will not be affected by the horizontal velocity:

$$y' = y$$
$$z' = z$$

But the x coordinates will be affected. We assume it will be a linear transformation:

$$x' = k(x - vt)$$
$$x = k'(x' + vt)$$

But in *Relativity* the transformation equations should have the same form (the laws of physics must be the same). Only the relative velocity matters. So $k' = k$.

Consider the outgoing light wave along the x-axis ($y = z = 0$).

$x' = ct'$ in frame S'

$x = ct$ in frame S

Now plug these into the transformation equations:

$x' = k(x - vt) = k(ct - vt) = kct(1 - v/c)$ and
$x = k(x' + vt') = k(ct' + vt') = kct'(1 + v/c)$

Plug these two equations into the light wave equation:

$$ct' = x' = kct(1 - v/c)$$
$$ct = x = kct'(1 + v/c)$$
$$t' = kt(1 - v/c)$$
$$t = kt'(1 + v/c)$$

Plug t' into the equation for t:

$$t = k^2 t(1 + v/c)(1 - v/c)$$
$$1 = k^2\left(1 - v^2/c^2\right)$$
$$\Rightarrow k = \frac{1}{\sqrt{1 - v^2/c^2}} \equiv \gamma$$

So the modified transformation equations for the spatial coordinates are:

$$x' = \gamma(x - vt)$$
$$y' = y$$
$$z' = z$$

Now what about time?

$x' = \gamma(x - vt)$
$x = \gamma(x' + vt')$ inverse transformation
Plug one into the other:
$x = \gamma(\gamma x - \gamma vt + vt')$

Solve for *t*’:

$$x-\gamma^2 x+\gamma^2 vt=\gamma vt'$$
$$x\left(1-\gamma^2\right)+\gamma^2 vt=\gamma vt'$$
$$x\frac{1-v^2/c^2-1}{1-v^2/c^2}+\gamma^2 vt=\gamma vt'$$
$$-\gamma^2 xv^2/c^2+\gamma^2 vt=\gamma vt'$$
$$\Rightarrow t'=\frac{1}{\gamma v}\left(-\gamma^2 xv^2/c^2+\gamma^2 vt\right)$$
$$t'=\gamma\left(t-vx/c^2\right)$$

So the correct transformation (and inverse transformation) equations are:

$$\boxed{\begin{array}{ll} x'=\gamma(x-vt) & x=\gamma(x'+vt') \\ y'=y & y=y' \\ z'=z & z=z' \\ t'=\gamma\left(t-vx/c^2\right) & t=\gamma\left(t'+vx'/c^2\right) \end{array}}$$

The Lorentz Transformation

Note the following features:

- We recover the Galilean transformation if $c\to\infty$ (or $v\to 0$) so that $\gamma\to 1$
- Space and time coordinates are mixed (x,t)
- No change in form of equations from one frame to another (Einstein’s 1st postulate)
- Only relative velocities matter

Also, note that you can derive time dilation and length contraction from these equations. For example, if a clock sits at rest in frame S’ at position $x'=0$, then an observer in frame S measures the period of the clock to be $T=\gamma\,\tau$.

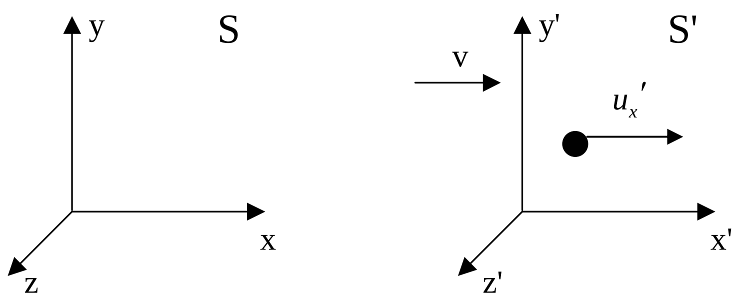
Moreover, note that two events which are simultaneous in frame S’ (say at time $t'=0$ and at positions x_1' and x_2') are not simultaneous in frame S ($t_1\neq t_2$).

Relativity 3

Disclaimer: These lecture notes are not meant to replace the course textbook. The content may be incomplete. Some topics may be unclear. These notes are only meant to be a study aid and a supplement to your own notes. Please report any inaccuracies to the professor.

Addition of Velocities

Now that we know that the Galilean transformation must be modified, it's time to revisit the topic of adding velocities. Consider two inertial frames S and S' with a relative velocity *v*.



$u_x' = \dfrac{dx'}{dt'}$ in frame S'

$u_x = u_x' + v$ in a Galilean transformation, which would imply

$u_x > c$ if $u_x' = c$

Consider the inverse Lorentz Transformation:

$x = \gamma(x' + vt')$ and $y = y', z = z'$

$t = \gamma\left(t' + vx'/c^2\right)$

Take differentials:

$dx = \gamma(dx' + vdt')$

$dt = \gamma\left(dt' + v/c^2 dx'\right)$

Divide one by the other:

$$u_x = \frac{dx}{dt} = \frac{dx' + v dt'}{dt' + v/c^2 dx'}$$

$$u_x = \frac{\frac{dx'}{dt'} + v}{1 + v/c^2 \frac{dx'}{dt'}}$$ where we have divided by dt'

Note that $\frac{dx'}{dt'} = u_x{}'$

The velocity addition formulae are:

$$\boxed{\begin{aligned} u_x &= \frac{u_x{}' + v}{1 + v u_x{}'/c^2} \\ u_y &= \frac{u_y{}'}{\gamma\left(1 + v u_x{}'/c^2\right)} \\ u_z &= \frac{u_z{}'}{\gamma\left(1 + v u_x{}'/c^2\right)} \end{aligned}}$$

Note that even though $y = y'$ and $z = z'$, that $u_y \neq u_y{}'$ and $u_z \neq u_z{}'$

Example: Consider a spacecraft that travels at 0.8*c* from Earth and that launches a projectile with a relative velocity of 0.8*c*. What is the velocity of the projectile from Earth?

Galilean: $u_x = 0.8c + 0.8c = 1.6c > c$!

Lorentz: $u_x = \frac{1.6c}{1 + 0.8^2} = 0.976c < c$

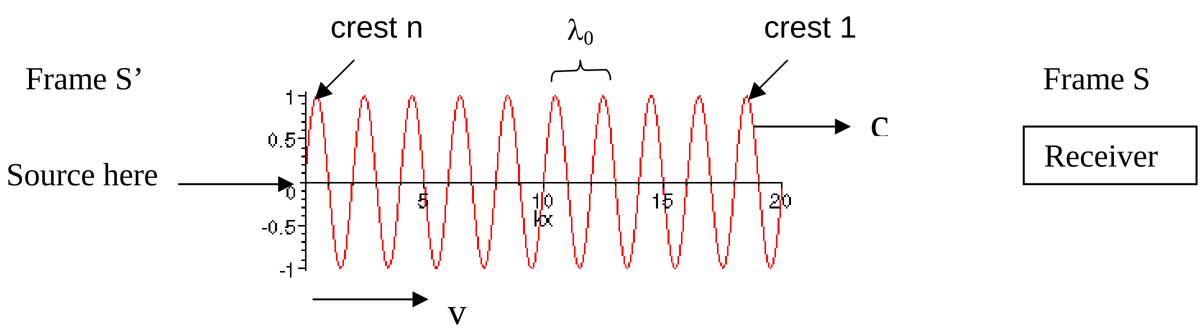
If instead of a projectile we turned on a light beam, both observers on the spacecraft and on Earth would agree that the velocity of the light beam is *c*, as required by Einstein's 2nd postulate.

The addition of velocity formulae tell us that nothing can exceed the speed of light.

Doppler Effect

The Doppler effect is a change in frequency of a traveling wave when the source is moving toward or away from a receiver, like the change in pitch of a car's engine when it travels by you. We can derive the change in "pitch" for light using what we have learned in Special Relativity.

Consider a light wave traveling along the x-axis. It emits n wave crests in a time T_0 in the rest frame of the emitter.



Length of wavetrain $= L_0 = n\lambda_0 = cT_0$

$$\Rightarrow \lambda_0 = \frac{cT_0}{n}$$

For light, $\lambda f = c$

$$\Rightarrow f_0 = \frac{c}{\lambda_0} = \frac{n}{T_0} = \text{ frequency of light in rest frame}$$

The frequency is n crests per time T_0

Now consider a receiver in a different inertial frame S. Suppose that the transmitter in frame S' is moving toward the receiver at a velocity v. Let's compute the frequency received given that the speed of light is always a constant for all frames.

$L = n\lambda = cT - vT$ length of wavetrain in frame S

Distance traveled by crest 1 minus distance source moves by the time of the last crest.

$$\Rightarrow \lambda = \frac{c-v}{n}T$$

Now from time dilation we know that $T = \gamma T_0$

$$\lambda = \frac{c-v}{n}\gamma T_0$$

$$\Rightarrow f = \frac{cn}{(c-v)\gamma T_0}$$

$$f = \frac{f_0}{\gamma(1-v/c)} \quad \text{Since } f_0 = n/T_0$$

Now we substitute in for γ:

$$f = \frac{\sqrt{1 - v^2/c^2}}{1 - v/c} f_0 = \frac{\sqrt{(1+v/c)(1-v/c)}}{1 - v/c} f_0$$

So the Doppler shift equations are:

$$\boxed{\begin{array}{ll} f = f_0\sqrt{\dfrac{1+v/c}{1-v/c}} & \text{Source and receiver approaching} \\ f = f_0\sqrt{\dfrac{1-v/c}{1+v/c}} & \text{Source and receiver receding} \end{array}}$$

Thus, when source and receiver approach each other, the frequency is shifted higher. We say that the light is **blue-shifted**.

When source and receiver recede from each other, the frequency is shifted lower. We say that the light is **red-shifted**. Red light has a lower frequency than blue light.

Modern electronics allow us to determine frequencies very accurately, so we can measure relative velocities accurately as well using this effect. Examples include Doppler weather radar, police radar, and even the expansion of the universe!

Lorentz Invariance

We have seen that some quantities change from one inertial frame to another (length, time, velocity, frequency). A quantity which does not change after a Lorentz transformation is said to be **Lorentz Invariant**. One special invariant is the

Space-time Interval:

$$(\Delta s)^2 = (c\Delta t)^2 - \left[(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2\right]$$
$$\Delta t = t_2 - t_1$$
$$\Delta x = x_2 - x_1 \text{ etc.}$$

This is the generalization of Cartesian distance for 4-dimensional space-time. The same value for Δs is obtained for any inertial frame. So although length and time separately are not invariant from one frame to another, this particular combination is.

We can prove that this is true by applying the Lorentz transformation. For example, consider a subatomic particle which decays in a time τ in its rest frame.

In the rest frame S':

$$t_1' = 0, \quad t_2' = \tau, \quad x_1' = x_2' = y_1' = \cdots = 0$$
$$\Rightarrow \Delta s = c\tau$$

Now make a Lorentz transformation to another frame S moving at velocity *v*:

$$x = \gamma(x' + vt') \qquad y' = y = z' = z = 0$$
$$t = \gamma\left(t' + vx'/c^2\right)$$
$$x_2 = \gamma v\tau, \quad x_1 = 0 \quad \Rightarrow \Delta x = \gamma v\tau$$
$$t_2 = \gamma\tau, \qquad t_1 = 0 \quad \Rightarrow \Delta t = \gamma\tau$$
$$(\Delta s)^2 = c^2\gamma^2\tau^2 - \gamma^2 v^2\tau^2 = \gamma^2\tau^2 c^2\left(1 - v^2/c^2\right)$$
$$= \frac{c^2\tau^2}{1 - v^2/c^2}\left(1 - v^2/c^2\right)$$
$$\Rightarrow \Delta s = c\tau \text{ as in the rest frame}$$

Some terminology:

$\Delta s^2 > 0 \quad \Rightarrow \quad$ time-like

A frame exists where 2 events occur in one place, separated by time.

$\Delta s^2 = 0 \quad \Rightarrow \quad$ light-like

2 events are separated by the speed of light.

$\Delta s^2 < 0 \quad \Rightarrow \quad$ space-like

No light signal can connect the 2 events.

Relativity 4

Disclaimer: These lecture notes are not meant to replace the course textbook. The content may be incomplete. Some topics may be unclear. These notes are only meant to be a study aid and a supplement to your own notes. Please report any inaccuracies to the professor.

Relativistic Momentum

Newton's 2nd Law can be written in the form

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}$$

where the non-relativistic momentum of a body is $\mathbf{p} = m\mathbf{u}$ where $\mathbf{u} = \frac{d\mathbf{x}}{dt}$. However, because of the Lorentz transformation equations, $\frac{d\mathbf{x}}{dt}$ is measured differently in different inertial frames. Thus, Newton's 2nd Law would not have the same form in different frames. We need a new definition of momentum to retain the definition of force as a change in momentum.

Suppose $\mathbf{p} = m\frac{d\mathbf{x}}{d\tau}$, where τ is the proper time in the object's rest frame. Every observer will agree on which frame is the rest frame. Also, since $y' = y$ and $z' = z$, the transverse momentum (p_y and p_z) will be invariant for a Lorentz transformation along the x axis. (This would not be the case if we did not use the proper time in the definition). We can rewrite this momentum definition as follows:

$$\mathbf{p} = m\frac{d\mathbf{x}}{d\tau} = m\frac{d\mathbf{x}}{dt}\frac{dt}{d\tau}$$

$$t = \gamma\,\tau \quad \Rightarrow \quad \frac{dt}{d\tau} = \gamma \qquad \text{From time dilation}$$

$$\boxed{\mathbf{p} = \gamma_u\, m\, \mathbf{u} \qquad \gamma_u = \frac{1}{\sqrt{1-u^2/c^2}}}$$

Recall that momentum is a vector quantity. Conservation of momentum, which still applies in Special Relativity, implies that each component of momentum is conserved.

Note that **u** is the velocity of the object in a reference frame, not the velocity of a reference frame relative to another.

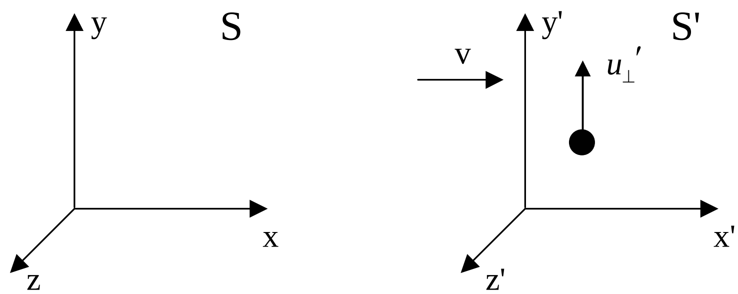
In this definition of momentum, the mass $m=m_0$ is the "rest mass". That is, it is the mass of an object in its rest frame. Sometimes γm is referred to as the "relativistic mass", such that we can retain the Newtonian definition of momentum as $\mathbf{p} = m\mathbf{u}$. In this sense, the mass of an object grows as its velocity increases. But this convenient trick can be problematic. As we shall see, the kinetic energy, for example, is not ½ mv^2.

Relativistic Force

With the previous relativistic definition for momentum, we can retain the usual definition for **force**:

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = \frac{d}{dt}\left(m\frac{d\mathbf{x}}{d\tau}\right) = \frac{d}{dt}\left(\gamma_u\ m\mathbf{u}\right) \qquad \text{where } \mathbf{u} = \frac{d\mathbf{x}}{dt} \quad \text{and} \quad \gamma_u = \frac{1}{\sqrt{1-u^2/c^2}}$$

It is useful to consider how force transforms under a Lorentz Transformation:



According to the addition of velocity formulae, the transformation of the velocity perpendicular to the direction of the Lorentz Transformation is:

$$\mathbf{u}_\perp = \frac{\mathbf{u}_\perp'}{\gamma_v\left(1+vu_x'/c^2\right)} \quad \text{where} \quad \gamma_v = \frac{1}{\sqrt{1-v^2/c^2}}$$

So for the perpendicular force, which can be written as:

$$\mathbf{F}_\perp = m\frac{d}{dt}\frac{d\mathbf{x}}{d\tau} = m\frac{d}{d\tau}\mathbf{u}_\perp$$

it transforms as:

$$\mathbf{F}_\perp = m\frac{d}{d\tau}\frac{\mathbf{u}_\perp'}{\gamma_v\left(1+vu_x'/c^2\right)} = \frac{1}{\gamma_v\left(1+vu_x'/c^2\right)}m\frac{d\mathbf{u}_\perp'}{d\tau}$$

$$\mathbf{F}_\perp = \frac{\mathbf{F}_\perp'}{\gamma_v\left(1+vu_x'/c^2\right)}$$

where we assume no acceleration in the direction parallel to the transformation

Relativistic Energy

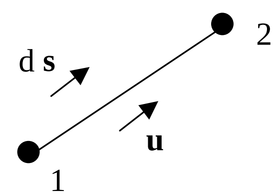
Now **work** is defined as force applied over a distance. It corresponds to the expended energy to accelerate a body. If the force and path are constant,

$$W = F \cdot d$$

More generally, if the force and path vary, then a line integral must be performed from initial position 1 to final position 2.

$$W_{12} = \int_1^2 \mathbf{F} \cdot d\mathbf{s}$$

The work applied to a body translates to a change in the kinetic energy since energy must be conserved. If we assume that the body is initially at rest, then the final kinetic energy is equal to the work expended:



$$W = K = \int \frac{d}{dt}(\gamma m\mathbf{u}) \cdot \mathbf{u}\, dt$$ where we have used $d\mathbf{s} = \mathbf{u}\, dt$

$$K = m\int dt \frac{d}{dt}(\gamma\mathbf{u}) \cdot \mathbf{u}$$

$$K = m\int_0^{\gamma U} u\, d(\gamma u)$$

Integrate by parts:

$$K = \gamma mU^2 - m\int_0^U \gamma u\, du$$

$$= \gamma mU^2 - m\int_0^U \frac{u\, du}{\sqrt{1-u^2/c^2}}$$

You can check this integral by differentiation

$$= \gamma mU^2 + mc^2\sqrt{1-u^2/c^2}\,\Big|_0^U$$

$$= \gamma mU^2 + mc^2\sqrt{1-U^2/c^2} - mc^2$$

$$= \gamma\left[mU^2 + mc^2\left(1-U^2/c^2\right)\right] - mc^2$$

Thus, we get for the relativistic kinetic energy:

$$\boxed{K = \gamma mc^2 - mc^2 = (\gamma - 1)mc^2}$$

This final expression for the kinetic energy looks like nothing like the non-relativistic equation $K = \frac{1}{2}mu^2$. However, if we consider velocities much less than the speed of light, we can see the correspondence:

$$\gamma = \left(1 - u^2 / c^2\right)^{-1/2} \approx 1 + \frac{1}{2} u^2 / c^2 + \cdots \quad \text{using the binomial expansion}$$

$$\Rightarrow K = (\gamma - 1)mc^2 \approx \frac{1}{2}\frac{u^2}{c^2} mc^2 = \frac{1}{2} mu^2 \quad \text{for } u << c$$

So at low velocities there is no difference between the definition of kinetic energy in Special Relativity from that in Newtonian Mechanics.

Now let's consider the opposite limit when the velocity approaches the speed of light. In that case, the kinetic energy becomes infinite as the relativistic factor γ goes to infinity. This is another way of saying that objects cannot exceed the speed of light, because it would take an infinite amount of energy.

Now let's rewrite the equation involving the kinetic energy:

$$\boxed{E \equiv \gamma mc^2 = K + mc^2}$$

This equation has the form of kinetic energy plus potential energy equals total energy. What is the potential energy? It is the term:

$$\boxed{E_0 = mc^2}$$

which we refer to as the **rest energy**. As you know, this is Einstein's famous equation that tells us that mass is another form of energy. Mass can be converted into energy and vice versa. How much energy? Let's see:

Example: Suppose that a 1 kg mass moves at a velocity u = 1 m/s. The kinetic energy is ½ $m\, u^2$ = ½ J. (We can use the non-relativistic equation because the velocity is much much smaller than the speed of light.) The rest mass energy is $mc^2 = 9.0 \times 10^{16}$ J. Clearly there is a tremendous amount of energy in 1 kg of mass. That is why nuclear weapons have the power that they do, because they convert a significant amount of mass into energy.

Conservation of Energy:

We have learned in earlier physics courses that kinetic energy does not have to be conserved in an inelastic collision. Likewise, mass does not have to be conserved since it can be converted into energy. However, the total energy (kinetic, rest mass, and all other potential energy forms) is always conserved in Special Relativity. Momentum and energy are conserved for both elastic and inelastic collisions when the relativistic definitions are used.

Relationship between Energy and Momentum

Using the **Newtonian** definitions of energy and momentum,
$E = \frac{1}{2}mu^2$ and $p = mu$, we can write:

$$\boxed{E = \frac{p^2}{2m}}$$

Now consider the **relativistic** definitions:

$$E = \gamma mc^2$$
$$p = \gamma mu$$
$$p^2 = \gamma^2 m^2 u^2$$
$$p^2c^2 = \gamma^2 m^2 u^2 c^2 = \gamma^2 m^2 c^4 \frac{u^2}{c^2} = \gamma^2 m^2 c^4 \left(1 - \frac{1}{\gamma^2}\right)$$
$$p^2c^2 = \gamma^2 m^2 c^4 - m^2 c^4$$

But $E = \gamma mc^2$

So $p^2c^2 = E^2 - m^2c^4$

Thus the equivalent relationship between energy and momentum in Relativity is:

$$\boxed{E^2 = p^2c^2 + m^2c^4 \quad \text{or equivalently} \quad m^2c^4 = E^2 - p^2c^2}$$

This is another example of **Lorentz Invariance**. No matter what inertial frame is used to compute the energy and momentum, $E^2 - p^2c^2$ always given the rest energy of the object. Energy and momentum take the role of time and space in the other Lorentz invariant quantity Δs. In fact, we refer to (t, x, y, z) and (E, p_x, p_y, p_z) as **four-vectors**, and the "lengths" of these vectors are these Lorentz-invariant expressions we derived.

Particles without mass are a special case

$$\Rightarrow E = pc$$

E and pc can also be written: $E = \gamma mc^2$ and $pc = \gamma muc$.
The only way we can reconcile these last two definitions with $E = pc$ is to set the velocity to c. **Massless particles must travel at the speed of light.**
As we will learn, light itself is composed of particles (photons). To travel at the speed of light, these particles must be massless.

The Electron-Volt Energy Unit

The Lorentz force law is $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$, where $\mathbf{E}$ is the electric field and $\mathbf{B}$ is the magnetic field. The work done to move a charged particle in an electric field only is:

$$W_{12} = \int_1^2 \mathbf{F} \cdot d\mathbf{s} = q\int_1^2 \mathbf{E} \cdot d\mathbf{s}$$
$$= q\left(V_2 - V_1\right)$$

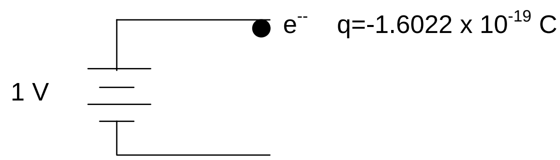
The electric potential is ϕ (such that the electric field $\mathbf{E} = -\nabla V$). We can summarize the work done by;

$$\boxed{W = q\Delta V \qquad \Delta V = \text{potential difference}}$$

Consider the work done to move an electron across a potential difference of 1 Volt?

$$W = \left(-1.6022 \times 10^{-19}\ \text{C}\right)\left(-1\ \text{V}\right) = 1.6022 \times 10^{-19}\ \text{J}$$

This is a very small unit! We define it as a new unit of energy, the **electron-volt**:



$$\boxed{1\ \text{eV} = 1.6022 \times 10^{-19}\ \text{J}}$$

Example: Express the electron rest mass energy in this new unit:

$$E_0 = m_e c^2 = \left(9.11 \times 10^{-31}\ \text{kg}\right)\left(3.0 \times 10^8\ \text{m / s}\right)^2 \frac{1\ \text{eV}}{1.6022 \times 10^{-19}\ \text{J}}$$
$$E_0 = 511{,}000\ \text{eV} \qquad (\text{or } 511\ \text{keV},\ 0.511\ \text{MeV},\ 0.000511\ \text{GeV})$$

We also can define new units for mass and momentum. For example, the mass of the electron can be expressed $m_e = 0.511\ \text{MeV} / c^2$. In other words, if you multiply the mass by c^2, you get the rest energy in electron-volts.

Similarly, we know that pc has units of energy, so momentum can be expressed in units like MeV / c. In other words, if you multiply by c, you get an energy in electron-volts.

Invariant Mass

We can now apply the relativistic definitions of energy and momentum to calculations of particle collisions. In particular, we can compute the rest mass of a particle formed when two particles annihilate into pure energy and then form a new particle.

Example: An electron and a positron (an anti-electron) annihilate with equal and opposite momentum: $p = 1.55 \text{ GeV}/c$. (Note the new momentum unit). The collision produces a new particle called the J/ψ in the following reaction: $e^- + e^+ \rightarrow J/\psi$. What is the mass of this new particle?

We need to compute the **invariant mass** of the electron-positron initial state to determine the rest mass of the new particle:

$Mc^2 = \sqrt{E_{tot}^2 - p_{tot}^2 c^2}$ where E_{tot} and p_{tot} are the total energy and momentum

$p_{tot} = p_1 + p_2 = 1.55 \text{ GeV}/c - 1.55 \text{ GeV}/c = 0$ by conservation of momentum

$E_{tot} = E_1 + E_2$ by conservation of energy

$$E_1 = \sqrt{p_1^2 c^2 + m^2 c^4} = \sqrt{(1.55 \text{ GeV})^2 + (0.000511 \text{ GeV})^2} \approx 1.55 \text{ GeV}$$

$E_1 = E_2$ because the magnitude of the momentum (and mass) is the same

$$\Rightarrow Mc^2 = E_{tot} = 1.55 + 1.55 \text{ GeV} = 3.1 \text{ GeV}$$

The J/ψ particle has a mass of 3.1 GeV/c. Note that we have made extensive use of the new Lorentz-invariant quantity involving energy and momentum

Binding Energy

As we have learned, mass is a form of potential energy. It can be converted into energy, or energy can be converted into mass. Because of this, **mass does not have to be conserved** in reactions. If you throw two balls at each other and they stick together (an **inelastic** collision), the resulting mass is not necessarily the sum of the individual masses of the two balls.

This surprising result makes sense when we consider that mass is just another form of potential energy. When two balls stick together, there must be some attractive force holding the composite system together. In the case of the hydrogen atom, an electron and proton are bound by an attractive electromagnetic force. To separate the electron and proton (*i.e.* ionize hydrogen), one must overcome the attractive force, and that takes energy. In other words, the particles have larger electromagnetic potential energy when separated than together. This potential energy is:

$$V = \frac{-e^2}{4\pi\varepsilon_0 r}$$

which increases as the separation distance r increases.

Where does this increase in energy go, since we know the total energy must be conserved? It goes into the rest mass energies of the electron and proton in the case of hydrogen. Another way of putting it is that the hydrogen atom has a mass that is **less** than the sum of the separate masses of the electron and proton. The difference in the rest mass energies of the separate objects from the combined system is called the **binding energy**:

$$BE = \left\{ M(\text{separate}) - M(\text{bound}) \right\} c^2$$

In the case of hydrogen, the binding energy is 13.6 eV; that is, hydrogen has a mass that is 13.6 eV less than the sum of the masses of the electron and proton.

Let's consider another example. The deuteron is a bound system of a neutron and a proton. The binding energy is given by:

$$BE = \left\{ M(\text{n}) + M(\text{p}) - M\left({}^2\text{H}\right) \right\} c^2$$

$$BE = \left\{ 939.57 \text{ MeV}/c^2 + 938.28 \text{ MeV}/c^2 - 1875.63 \text{ MeV}/c^2 \right\}$$

$$BE = 2.22 \text{ MeV}$$

Clearly nuclear binding energies are much larger than atomic binding energies! We will explore this more when we study nuclear physics toward the end of this course.

Reaction Energy

Closely related to binding energy is the concept of reaction energy. Not all composite systems have a mass less than the sum of its constituent masses, and some fundamental particles spontaneously decay into particles whose combined mass is less than that of the parent. In these cases, energy is released in the decay or reaction because of the difference in rest mass energies. We define this reaction energy as:

$$Q = \left\{ M(\text{initial products}) - M(\text{final products}) \right\} c^2$$

As you can see, it is just the negative of the binding energy. If Q is positive, we say that the reaction or decay is **exothermic**; that is, it releases energy. If Q is negative, the reaction or decay is **endothermic**; it takes energy to make it happen.

Example: Consider the spontaneous decay of a neutron: $n \to p + e^- + \nu_e$. We can calculate the energy released in this decay by taking the difference in mass of the left-hand side from the right-hand side. The neutrino (ν_e) will be discussed later in the nuclear and particle physics sections; what is relevant here is that its mass is essentially zero.

$$Q = \left\{ M(n) - M(p) - M(e^-) \right\} c^2$$

$$Q = \left\{ 939.57 \text{ MeV} / c^2 - 938.28 \text{ MeV} / c^2 - 0.511 \text{ MeV} / c^2 \right\} c^2$$

$$Q = 0.78 \text{ MeV}$$