A truely “educational” treatment of the quasistationary limit

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1 Introduction

Im referring to the paper [BCG13a] and the comment and corrigendum to it [Her14, BCG13b]. While the starting point of the original paper is clearly wrong, as stated in Heras's comment, the corrected version is formally correct but misleading for students. This is related to the traditional way to introduce the “displacement current” as Maxwell's correction of Ampère's Law, following the historical development of electromagnetic theory in the 19^{th} century. Already this ansatz is, in my opinion, teaching an outdated and even wrong picture as if Maxwell's additional term in Ampère's Law is indeed an “additional current density” and thus a source of the magnetic field.

The modern understanding is due to the special theory of relativity, which has been found by Einstein in 1905 (after other earlier work by Poincaré, FitzGerald, and Lorentz) and brought to its final four-dimensional form by Minkowski in 1908, in studying the symmetry properties of Maxwell's equations with regard to the special principle of relativity (indistinguishability and physical equivalence of different inertial frames).

The following is far from new. It can be found in any decent textbook on classical electrodynamics, e.g., [Som01].

2 Retarded potentials and fields

Classical electrodynamics thus should be presented as a relativistic covariant local field theory, starting in the local form of Maxwell's equations in vacuo, written as (using Heaviside-Lorentz units for convenience)

$$\vec{\nabla} \cdot \vec{B} = 0, \tag{1}$$

$$\vec{\nabla} \times \vec{E} + \frac{1}{c}\partial_t \vec{B} = 0, \tag{2}$$

$$\vec{\nabla} \times \vec{B} - \frac{1}{c}\partial_t \vec{E} = \frac{1}{c}\vec{j}, \tag{3}$$

$$\vec{\nabla} \cdot \vec{E} = \rho. \tag{4}$$

Here, the structure of Maxwell's equations becomes very clear: Eqs. (1) and (2), the homogenous Maxwell equations, are constraint equations for the field components. They thus should be used to introduce the electromagnetic potentials as an ansatz to get rid of these constraints to simplify the solution of the equations of motion (3) and (4), the inhomogeneous Maxwell equations, that clearly identify the current and charge densities as the source of the electromagnetic field.

As is well known from Helmholtz's fundamental theorem of vector analysis, from (1) it follows the existence of a vector potential for the magnetic field,

$$\vec{B} = \vec{\nabla} \times \vec{A}, \tag{5}$$

and using this in (2) leads to

$$\vec{\nabla} \times \left(\vec{E} + \frac{1}{c} \partial_t \vec{A} \right) = 0, \tag{6}$$

and thus, again according to Helmholtz's theorem the existence of a scalar potential for the field in the brackets, i.e.,

$$\vec{E} = -\frac{1}{c} \partial_t \vec{A} - \vec{\nabla} \Phi. \tag{7}$$

The vector field is only determined up to a gradient field, and thus if $\vec{A}$ and Φ lead to the electromagnetic field $(\vec{E}, \vec{B})$ according to (5) and (7) this is also the case for

$$\vec{A}' = \vec{A} - \vec{\nabla} \chi, \quad \Phi' = \Phi + \frac{1}{c} \partial_t \chi. \tag{8}$$

This is the gauge invariance of Maxwell's electromagnetics. Thus we can use one "gauge constraint" on the potential $(\Phi, \vec{A})$ to simplify the equations of motion, which follow by plugging in (5) and (7) into the inhomogeneous Maxwell equations (3) and (4):

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) + \frac{1}{c^2} \partial_t^2 \vec{A} + \frac{1}{c} \partial_t \vec{\nabla} \Phi = \vec{j}, \quad -\frac{1}{c} \partial_t \vec{\nabla} \cdot \vec{A} - \vec{\nabla}^2 \Phi = \rho. \tag{9}$$

Using

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \vec{\nabla}^2 \vec{A}, \tag{10}$$

one finds from the first equation that the Lorenz-gauge constraint

$$\frac{1}{c} \partial_t \Phi + \vec{\nabla} \cdot \vec{A} = 0 \tag{11}$$

eliminates Φ from the equation and separates also the components of $\vec{A}$, leading to the wave equation

$$\Box \vec{A} = \vec{j} \quad \text{with} \quad \Box = \frac{1}{c^2} \partial_t^2 - \vec{\nabla}^2. \tag{12}$$

It also eliminates $\vec{A}$ from the second equation (9), leading also to the wave equation for Φ:

$$\Box \Phi = \rho. \tag{13}$$

Of course, (12) and (13) are only compatible with the Lorenz-gauge constraint (11), if electric charge is conserved, i.e.,

$$\partial_t \rho + \vec{\nabla} \cdot \vec{j} = 0. \tag{14}$$

This is, however, also a compatibility constraint for the Maxwell equations (1-4) themselves.

Now it's easy to solve the inhomogeneous wave equation, using the retarded propagator of the (1+3)-dimensional wave equation,

$$\begin{aligned}\Phi(t,\vec{x}) &= \frac{c}{4\pi}\int_{\mathbb{R}^4} \mathrm{d}t'\mathrm{d}^3\vec{x}\,'\frac{\delta[c(t-t')-|\vec{x}-\vec{x}\,'|]}{|\vec{x}-\vec{x}\,'|}\rho(t',\vec{x}\,') \\ &= \frac{1}{4\pi}\int_{\mathbb{R}^3}\mathrm{d}^3\vec{x}\,'\frac{1}{|\vec{x}-\vec{x}\,'|}\rho\left(t-\frac{|\vec{x}-\vec{x}\,'|}{c},\vec{x}\right), \\ \vec{A}(t,\vec{x}) &= \frac{1}{4\pi}\int_{\mathbb{R}^4} \mathrm{d}t'\mathrm{d}^3\vec{x}\,'\frac{\delta[c(t-t')-|\vec{x}-\vec{x}\,'|]}{|\vec{x}-\vec{x}\,'|}\vec{j}(t',\vec{x}\,') \\ &= \frac{1}{4\pi c}\int_{\mathbb{R}^3}\mathrm{d}^3\vec{x}\,'\frac{1}{|\vec{x}-\vec{x}\,'|}\vec{j}\left(t-\frac{|\vec{x}-\vec{x}\,'|}{c},\vec{x}\right).\end{aligned} \tag{15}$$

It is easy to show that these solutions indeed fulfill the Lorenz-gauge constraint (11).

The corresponding fields are found by using (5) and (7). To take the derivatives, it's most convenient to use the expressions with the δ distributions, making use of

$$\vec{\nabla}\delta[c(t-t')-|\vec{x}-\vec{x}\,'|] = \frac{\vec{x}-\vec{x}\,'}{|\vec{x}-\vec{x}'|}\frac{1}{c}\partial_t\delta[c(t-t')-|\vec{x}-\vec{x}\,'|] = -\frac{\vec{x}-\vec{x}\,'}{|\vec{x}-\vec{x}'|}\frac{1}{c}\partial_{t'}\delta[c(t-t')-|\vec{x}-\vec{x}\,'|]. \tag{16}$$

This leads, after some manipulations to the retarded fields,

$$\begin{aligned}\vec{E}(t,\vec{x}) &= \frac{1}{4\pi}\int_{\mathbb{R}^3}\mathrm{d}^3\vec{x}\,'\left\{(\vec{x}-\vec{x}\,')\left[\frac{\rho(t_{\text{ret}},\vec{x}\,')}{|\vec{x}-\vec{x}\,'|^3}+\frac{\dot{\rho}(t_{\text{ret}},\vec{x}\,')}{c|\vec{x}-\vec{x}\,'|^2}\right]-\frac{\dot{\vec{j}}(t_{\text{ret}},\vec{x}\,')}{c^2|\vec{x}-\vec{x}\,'|}\right\}, \\ \vec{B}(t,\vec{x}) &= \frac{1}{4\pi}\int_{\mathbb{R}^3}\mathrm{d}^3\vec{x}\,'\left[\frac{\vec{j}(t_{\text{ret}},\vec{x}\,')}{c|\vec{x}-\vec{x}\,'|^3}+\frac{\dot{\vec{j}}(t_{\text{ret}},\vec{x}\,')}{c^2|\vec{x}-\vec{x}\,'|^2}\right]\times(\vec{x}-\vec{x}\,'),\end{aligned} \tag{17}$$

which equations are known also as Jefimenko equations. The quasi-stationary limit is then obtained by a formal expansion of the retarded sources around the instant time in powers of $1/c$. From now on we use the abbreviation

$$t_{\text{ret}} = t - \frac{|\vec{x}-\vec{x}\,'|}{c} \tag{18}$$

for the retarded-time argument in convolutions of functions with the retarded Green's function.

One should emphasize in teaching classical electrodynamics that (17) clearly shows that a time-varying electric field or the "displacement current" $\partial_t\vec{E}$ is **not** a causal source of the magnetic field, as is clearly shown in (17). To the contrary, the causal source of the magnetic field is solely the **electric current density** (consisting of moving charges)! The same holds true for the electric field, which of course also contains contributions from the charge density in addition to the one from the time-varying current density.

Now, to find the quasi-stationary approximation we note that for any "retarded function" f we obtain

$$f(t_{\text{ret}},\vec{x}) = f(t,\vec{x}) - \frac{|\vec{x}-\vec{x}\,'|}{c}\dot{f}(t,\vec{x}) + \mathscr{O}\left[\ddot{f}(t,\vec{x})\left(\frac{|\vec{x}-\vec{x}\,'|}{c}\right)^2\right]. \tag{19}$$

For the magnetic field this leads to

$$\vec{B}(t,\vec{x}) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \mathrm{d}^3\vec{x}\,' \left\{\vec{j}(t,\vec{x}\,') + \mathscr{O}\left[\ddot{\vec{j}}(t,\vec{x}\,')\left(\frac{|\vec{x}-\vec{x}\,'|}{c}\right)^2\right]\right\} \times \frac{(\vec{x}-\vec{x}\,')}{c|\vec{x}-\vec{x}\,'|^3}. \tag{20}$$

For the electric field we find

$$\begin{aligned}\vec{E}(t,\vec{x}) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \mathrm{d}^3\vec{x}\,' \Bigg\{ &\frac{\vec{x}-\vec{x}\,'}{|\vec{x}-\vec{x}\,'|^3}\left[\rho(t,\vec{x}\,') + \mathscr{O}\left(\ddot{\rho}(t,\vec{x}\,')\frac{|\vec{x}-\vec{x}\,'|^2}{c^2}\right)\right] \\ &+ \frac{1}{|\vec{x}-\vec{x}\,'|^3}\mathscr{O}\left(\dot{\vec{j}}(t,\vec{x}\,')\frac{|\vec{x}-\vec{x}\,'|^2}{c^2}\right)\Bigg\}.\end{aligned} \tag{21}$$

The validity range for this expression is most easily understood when we assume a harmonic time dependence of the sources $\rho, \vec{j} \propto \exp(-\mathrm{i}\omega t)$. Then the leading neglected term in the bracket under the integral for the magnetic field becomes of the order $\mathscr{O}(\omega^2|\vec{x}-\vec{x}\,'|^2/c^2) = \mathscr{O}(|\vec{x}-\vec{x}\,'|^2/\lambda^2)$. The same holds for the first correction in the first line of (21). The second correction with $\dot{\vec{j}}$ can be estimated as of order $|x-\vec{x}\,'|/c$ and then using $\vec{j} = \rho\vec{v}_{\text{drift}}$ also smaller by and additional factor of $\mathscr{O}(|\vec{v}_{\text{drift}}|/c)$, supposed the drift velocity is small compared to the speed of light which is always the case for usual "house-hold currents".

This finally implies that the quasistationary approximation for the fields in free space is good in the region close to the sources, where the retardation effect of the finite speed of light doesn't play a role. Of course the same argument holds for the usual application of the quasi-stationary limit in circuit theory, where the extension of the circuit is small compared to the "typical wavelength" of the electromagnetic fields.

3 Direct approach without potentials

We can also come to the Jefimenko equations and the quasistationary limit by considering the Maxwell equations of the fields without introducing the potentials. We start with the magnetic field, taking the curl of (3) and using (1) and (2) in the corresponding expression to derive the wave equation for the magnetic field as

$$\Box\vec{B} = \frac{1}{c}\vec{\nabla}\times\vec{j}. \tag{22}$$

Using again the retarded Green's function leads to the solution

$$\vec{B}(t,\vec{x}) = \frac{1}{4\pi}\int_{\mathbb{R}^4} \mathrm{d}t'\mathrm{d}^3\vec{x}\,' \frac{\delta[c(t-t')-|\vec{x}-\vec{x}\,'|]}{|\vec{x}-\vec{x}\,'|}\,\vec{\nabla}\,'\times\vec{j}(t',\vec{x}\,'). \tag{23}$$

To show that this is identical with (17) for current densities that vanish sufficiently fast at infinity we note that by integration by parts we have

$$\int_{\mathbb{R}^4} \mathrm{d}t'\mathrm{d}^3\vec{x}\,' \frac{\delta[\ldots]}{|\vec{x}-\vec{x}\,'|}\vec{\nabla}' f(t',\vec{x}') = -\int_{\mathbb{R}^4} \mathrm{d}t'\mathrm{d}^3\vec{x}\,' f(t',\vec{x}\,')\vec{\nabla}'\left(\frac{\delta[\ldots]}{|\vec{x}-\vec{x}\,'|}\right), \tag{24}$$

where $\delta(\ldots)$ is always the expression in (23). Evaluating the derivative in (23) one finds after some simple manipulations

$$\int_{\mathbb{R}^4} \mathrm{d}t'\mathrm{d}^3\vec{x}\,' \frac{\delta[\ldots]}{|\vec{x}-\vec{x}\,'|}\vec{\nabla}' f(t',\vec{x}') = -\frac{1}{c}\int_{\mathbb{R}^3} \mathrm{d}^3\vec{x}\,'\left[f(t_{\text{ret}},\vec{x}\,') + \dot{f}(t_{\text{ret}},\vec{x}\,')\frac{|\vec{x}-\vec{x}\,'|}{c}\right]. \tag{25}$$

Using this rule in (23) we recover indeed (17) as it should be.

For the electric field, one takes the curl of (2) and then uses (3) and (4) to derive the wave equation

$$\Box \vec{E} = -\vec{\nabla}\rho - \frac{1}{c^2}\partial_t \vec{j}. \tag{26}$$

This gives

$$\vec{E}(t,\vec{x}) = -\frac{c}{4\pi}\int_{\mathbb{R}^4} \mathrm{d}t' \mathrm{d}^3\vec{x}\,' \left[\vec{\nabla}'\rho(t',\vec{x}\,') + \frac{1}{c^2}\partial_{t'}\vec{j}(t,\vec{x}\,')\right] \frac{\delta(\ldots)}{|\vec{x}-\vec{x}\,'|}. \tag{27}$$

Using again (24) for the $\vec{\nabla}'\rho$ term again leads (17). We note that (26) is only compatible with Gauss's Law (4), if the continuity equation for the electric charge (14) is fulfilled, as is already found above as an integrability constraint for the inhomogeneous Maxwell equations (3) and (4).

4 Quasistationary limit from Maxwell's equations

The quasistationary limit is more easily obtained directly from Maxwell's equations either. This also sheds some light on the fact that Maxwell's addition of the "displacement current" is responsible for the retardation effect, i.e., the finite propagation speed of electromagnetic waves, i.e., the speed of light.

Indeed, neglecting the "displacement current" in the Ampère-Maxwell Law (3) leads back to the original Ampère Law,

$$\vec{\nabla} \times \vec{B} = \frac{\vec{j}}{c}. \tag{28}$$

One should note in the first place that this is inconsistent with the continuity equation for the electric charge (14). Thus, for the quasitationary limit to be valid we must be able to neglect $\partial_t \rho$ against $\vec{\nabla} \cdot \vec{j}$, which latter quantity vanishes by virtue of (28).

Using (1) from taking the curl of (28) we get the Poisson equation

$$\Delta \vec{B} = -\frac{1}{c}\vec{\nabla} \times \vec{j}, \tag{29}$$

and using the Green's function of the Laplacian leads to the Biot-Savart Law in the form

$$\vec{B}(t,\vec{x}) = \frac{1}{4\pi c}\int_{\mathbb{R}^3} \frac{\vec{\nabla}' \times \vec{j}(t,\vec{x}\,')}{|\vec{x}-\vec{x}\,'|}, \tag{30}$$

which can be brought into the usual form by the same manipulations that lead from (23) back to the Jefimenko expression (17),

$$\vec{B}(t,\vec{x}) = \frac{1}{4\pi c}\int_{\mathbb{R}^3} \mathrm{d}^3\vec{x}\,' \vec{j}(t,\vec{x}\,') \times \frac{\vec{x}-\vec{x}\,'}{|\vec{x}-\vec{x}\,'|}, \tag{31}$$

which indeed coincides with the neglect of higher-order retardation effects, cf. (20). One should, however, note that it is much harder to derive the validity conditions for the quasitationary approximation with this approach. Also the negligence of retardation effects does not become explicit here.

In conclusion one can say that the somewhat more complicated approach to the quasistationary limit of the Maxwell equation becomes better understandable, using the retarded solutions of the full Maxwell equations either via the potentials in Lorenz gauge or directly the electric and magnetic fields as shown above.

A Derivation of the retarded propagator

For convenience, here we derive the expression for the Green's function of the D'Alembert operator, $\Box = 1/c^2 \partial_t^2 - \Delta$, defined by

$$\Box G_{\text{ret}}(t,\vec{x}) = \delta(t)\delta^{(3)}(\vec{x}), \quad G_{\text{ret}}(t,\vec{x}) \propto \Theta(t). \tag{32}$$

The most convenient way to find an explicit solution is to introduce the "Mills representation", i.e., the Fourier transform of the Green's function with respect to position,

$$G(t,\vec{x}) = \int_{\mathbb{R}^3} \mathrm{d}^3\vec{k} \frac{\exp(\mathrm{i}\vec{k}\cdot\vec{x})}{(2\pi)^3} \tilde{G}(t,\vec{k}). \tag{33}$$

Plugging this into (32) one finds

$$\left(\frac{1}{c^2}\partial_t^2 + \vec{k}^2\right)\tilde{G}(t,\vec{k}) = \delta(t). \tag{34}$$

This is easily solved with the ansatz

$$\tilde{G}(t,\vec{k}) = \Theta(t) g(t,k), \quad k = |\vec{k}|, \quad g(0^+,k) = 0, \quad \partial_t g(0^+,k) = c^2, \tag{35}$$

leading to

$$\tilde{G}(t,\vec{k}) = \frac{c}{k}\sin(ckt). \tag{36}$$

The Fourier transformation (33) is easily performed in spherical coordinates with the polar axis in direction of $\vec{x}$, with $u = \cos\vartheta$ leading to

$$\begin{aligned}
G(t,\vec{x}) &= \frac{c\Theta(t)}{4\pi^2}\int_0^\infty \mathrm{d}k \int_{-1}^1 \mathrm{d}u\, ck\sin(kct)\exp(\mathrm{i}kru) \\
&= \frac{c\Theta(t)}{2\pi^2 r}\int_0^\infty \mathrm{d}k \sin(kct)\sin(kr) \\
&= \frac{c\Theta(t)}{4\pi^2 r}\int_0^\infty \mathrm{d}k \{\cos[k(r-ct)] - \cos[k(r+ct)]\} \\
&= \frac{c\Theta(t)}{8\pi^2 r}\int_{\mathbb{R}} \mathrm{d}k \{\cos[k(r-ct)] - \cos[k(r+ct)]\} \\
&= \frac{c\Theta(t)}{4\pi}\delta(ct-r) = \frac{\Theta(t)}{4\pi}\delta\left(t-\frac{r}{c}\right).
\end{aligned} \tag{37}$$

References

[BCG13a] S. Barbieri, M. Cavinato and M. Gilberti, An educational path for the magnetic vector potential and its physical implications, Eur. J. Phys. **34**, 1209 (2013), `http://stacks.iop.org/0143-0807/34/i=5/a=1209`.

[BCG13b] S. Barbieri, M. Cavinato and M. Gilberti, Corrigendum: An educational path for the magnetic vector potential and its physical implications, Eur. J. Phys. **35**, 029501 (2013), `http://dx.doi.org/10.1088/0143-0807/35/2/029501`.

[Her14] J. A. Heras, Comment on 'An educational path for the magnetic vector potential and its physical implications', Eur. J. Phys. **35**, 028001 (2014), `http://stacks.iop.org/0143-0807/35/i=2/a=028001`.

[Som01] A. Sommerfeld, **Vorlesungen über Theoretische Physik III, Elektrodynamik**, Verlag Harri Deutsch, Frankfurt/M. (2001).