Review Questions

Problem 5

Find the length of the arc determined by $x^2 = y^3$ from (0,0) to (1,1).

$$x^2 = y^3$$
$$\sqrt[3]{x^2} = \sqrt[3]{y^3}$$
$$y = f(x) = x^{\frac{2}{3}}$$

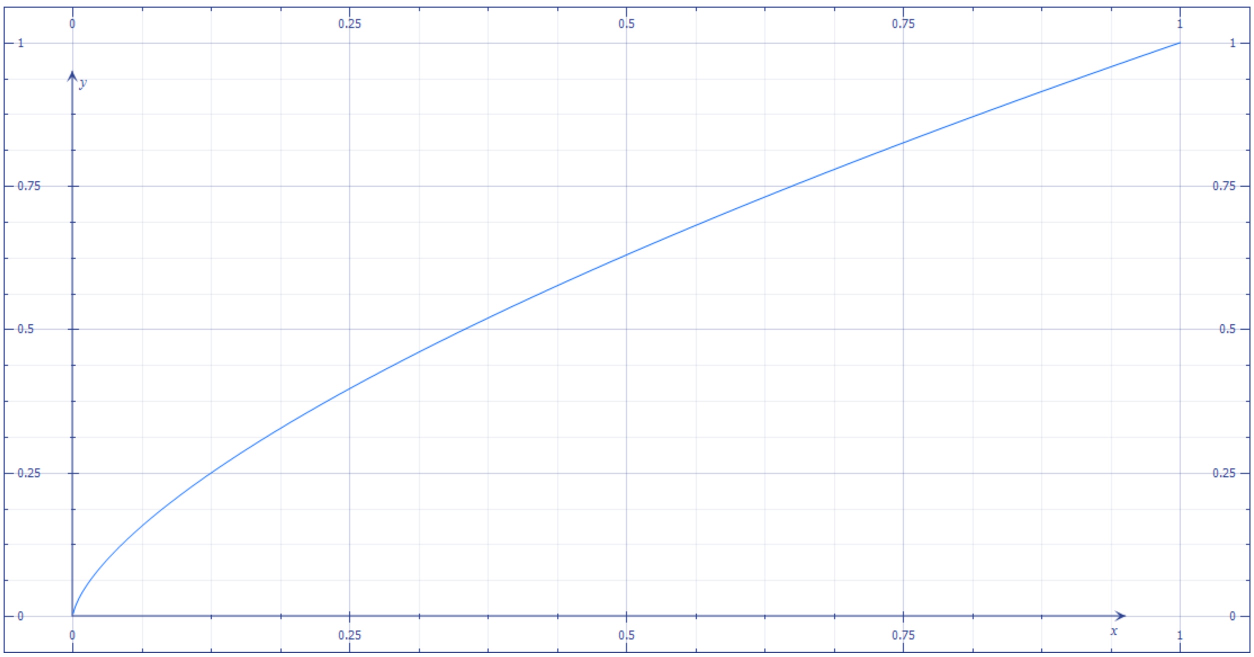

Figure 1

Arc length Definition

$$s = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

Set up

a=0, b=1 (as given by the problem)

$f(x) = x^{\frac{2}{3}}$, so $f'(x) = \frac{2}{3}x^{-\frac{1}{3}}$, and $[f'(x)]^2 = (\frac{2}{3}x^{-\frac{1}{3}})^2 = \frac{4}{9}x^{\frac{-2}{3}}$

Evaluate

$$s = \int_0^1 \sqrt{1 + \frac{4}{9} x^{\frac{-2}{3}}}\, dx$$

$$s = \int_0^1 \left(1 + \frac{4}{9} x^{\frac{-2}{3}}\right)^{\frac{1}{2}} dx$$

$$s = \int_0^1 u^{1/2}\, du \left(u = 1 + \frac{4}{9} x^{\frac{-2}{3}}\right)$$

$$\int u^{1/2}\, du = \frac{u^{3/2}}{\frac{3}{2}} + C = \left.\frac{u^{3/2}}{\frac{3}{2}}\right|_0^1$$

$$\left.\frac{\left(1 + \frac{4}{9} x^{\frac{-2}{3}}\right)^{3/2}}{\frac{3}{2}}\right|_0^1$$

$$1^{3/2} = 1$$

$$\left(\frac{4}{9} x^{\frac{-2}{3}}\right)^{3/2} = \left(\frac{4}{9}\right)^{3/2} \left(x^{\frac{-2}{3}}\right)^{3/2} = \left(\frac{\sqrt{4^3}}{\sqrt{9^3}}\right)\left(x^{\left(-2/3 \cdot 3/2\right)}\right) = \left(\frac{\sqrt{64}}{\sqrt{729}}\right)\left(x^{-6/6=-1}\right) = \frac{8}{27x}$$

$$\left.\frac{8}{27x} + \frac{2}{3}\right|_0^1 = \left(\frac{8}{27(1)} + \frac{2}{3}\right) - \left(\frac{8}{27(0)} + \frac{2}{3}\right) = \left(\frac{8}{27} + \frac{2}{3}\right) - \left(0 + \frac{2}{3}\right) = \frac{8}{27}$$