Then, providing the factor in brackets does not vanish, for which the required condition is easily shown to be $B^2 \neq 4AC$, we obtain

$$\frac{\partial^2 u}{\partial \zeta \partial \eta} = 0,$$

which has the successive integrals

$$\frac{\partial u}{\partial \eta} = F(\eta), \quad u(\zeta, \eta) = f(\eta) + g(\zeta).$$

This solution is just the same as (20.22),

$$u(x, y) = f(x + \lambda_2 y) + g(x + \lambda_1 y).$$

If the equation is parabolic (i.e. $B^2 = 4AC$), we instead use the new variables

$$\zeta = x + \lambda y, \qquad \eta = x,$$

and recalling that $\lambda = -(B/2C)$ we can reduce (20.20) to

$$A\frac{\partial^2 u}{\partial \eta^2} = 0.$$

Two straightforward integrations give as the general solution

$$u(\zeta, \eta) = \eta g(\zeta) + f(\zeta),$$

which in terms of $x$ and $y$ has exactly the form of (20.25),

$$u(x, y) = xg(x + \lambda y) + f(x + \lambda y).$$

Finally, as hinted at in subsection 20.3.2 with reference to first-order linear PDEs, some of the methods used to find particular integrals of linear ODEs can be suitably modified to find particular integrals of PDEs of higher order. In simple cases, however, an appropriate solution may often be found by inspection.

►*Find the general solution of*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6(x + y).$$

Following our previous methods and results, the complementary function is

$$u(x, y) = f(x + iy) + g(x - iy),$$

and only a particular integral remains to be found. By inspection a particular integral of the equation is $u(x, y) = x^3 + y^3$, and so the general solution can be written

$$u(x, y) = f(x + iy) + g(x - iy) + x^3 + y^3. ◄$$