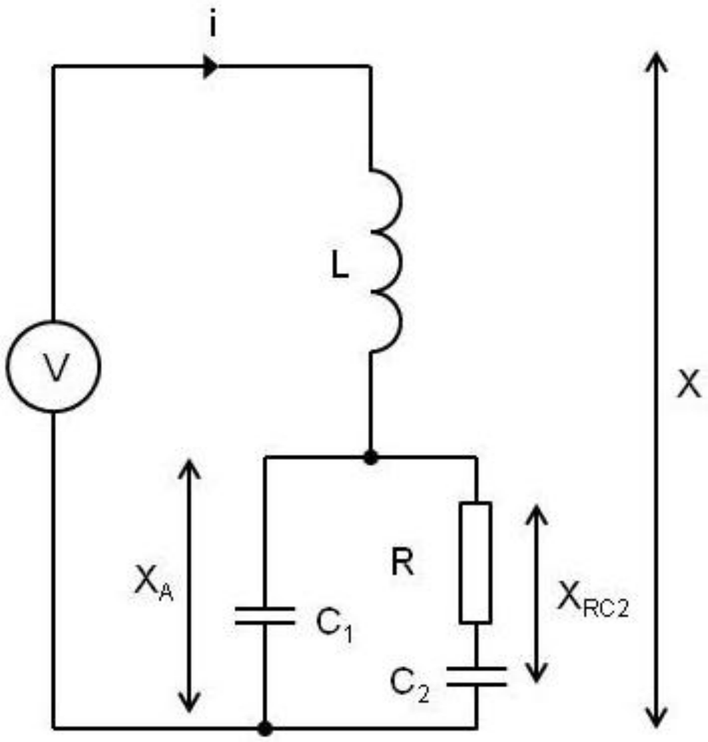
Introduction.

Goal: to find a general expression for the current "i" when the circuit below is subjected to a step change in voltage at t=0.



Let: X_{RC2} be the impedance of the series combination R plus C_2.
X_A be the impedance of X_{RC2} in parallel with C_1.
X be the impedance of X_A in series with L i.e. the total circuit impedance.

Intended procedure:

1. Determine an expression for impedance X in the s-domain.
2. Determine an expression for i(s), where i(s) = V/sX.
3. Re-arrange terms as necessary.
4. Transform back to the time domain, giving an expression for "i" in terms of t.

I will refer to the table of Laplace transforms at this link:

http://www.me.unm.edu/~starr/teaching/me380/Laplace.pdf

I have numbered each individual step in my working, in order to facilitate discussion.

1. Determine an expression for impedance X in the s-domain.

I have broken this step down into several stages. First, find X_{RC2}.

$$X_{RC2} = R + \frac{1}{sC_2} = \frac{1 + sRC_2}{sC_2} \qquad (1)$$

Next, find X_A.

$$X_A = \left(\frac{1}{X_{C1}} + \frac{1}{X_{RC2}}\right)^{-1} = \left(\frac{X_{RC2} + X_{C1}}{X_{C1}.X_{RC2}}\right)^{-1} = \frac{X_{C1}.X_{RC2}}{X_{RC2} + X_{C1}} \qquad (2)$$

$$= \frac{\frac{1}{sC_1}\cdot\frac{1+sRC_2}{sC_2}}{\frac{1+sRC_2}{sC_2} + \frac{1}{sC_1}} = \frac{\frac{1+sRC_2}{s^2C_1C_2}}{\frac{C_1(1+sRC_2)+C_2}{sC_1C_2}} \qquad (3)$$

$$= \frac{1+sRC_2}{s^2C_1C_2}\cdot\frac{sC_1C_2}{[C_1(1+sRC_2)+C_2]} \qquad (4)$$

$$= \frac{(1+sRC_2)C_1C_2}{s(C_1C_2)[C_1(1+sRC_2)+C_2]} \qquad (5)$$

$$= \frac{C_1C_2 + sRC_1C_2^2}{s(C_1C_2)[C_1 + sRC_1C_2 + C_2]} \qquad (6)$$

$$= \frac{C_1C_2 + sRC_1C_2^2}{sC_1^2C_2 + s^2RC_1^2C_2^2 + sC_1C_2^2} \qquad (7)$$

divide top and bottom by C_1:

$$= \frac{C_2 + sRC_2^2}{sC_1C_2 + s^2RC_1C_2^2 + sC_2^2} \qquad (8)$$

divide top and bottom by C_2:

$$= \frac{1+sRC_2}{sC_1 + s^2RC_1C_2 + sC_2} \qquad (9)$$

$$X_A = \frac{1+sRC_2}{s^2RC_1C_2 + s(C_1+C_2)} \qquad (10)$$

Now add the impedance of L (=sL) in series, to give the total impedance X:

$$X = \frac{1+sRC_2}{s^2RC_1C_2 + s(C_1+C_2)} + sL \qquad (11)$$

$$\boxed{X = \frac{1+sRC_2 + sL(s^2RC_1C_2 + s[C_1+C_2])}{s^2RC_1C_2 + s(C_1+C_2)}} \qquad (12)$$

2. Determine an expression for i(s).

Having defined the impedance for the whole network in the s-domain, now use this to find an expression for i(s).

$$i(s) = \frac{V}{sX} \qquad (13)$$

$$= V.\frac{1}{s\left(\dfrac{1 + sRC_2 + sL(s^2RC_1C_2 + s[C_1 + C_2])}{s^2RC_1C_2 + s(C_1 + C_2)}\right)} \qquad (14)$$

$$= V.\frac{s^2RC_1C_2 + s(C_1 + C_2)}{s(1 + sRC_2 + sL(s^2RC_1C_2 + s[C_1 + C_2]))} \qquad (15)$$

$$= V.\frac{sRC_1C_2 + (C_1 + C_2)}{1 + sRC_2 + sL(s^2RC_1C_2 + s[C_1 + C_2])} \qquad (16)$$

$$= V.\frac{sRC_1C_2 + (C_1 + C_2)}{1 + sRC_2 + s^3LRC_1C_2 + s^2LC_1 + s^2LC_2} \qquad (17)$$

$$= V.\frac{sRC_1C_2 + (C_1 + C_2)}{s^3LRC_1C_2 + s^2L(C_1 + C_2) + sRC_2 + 1} \qquad (18)$$

$$= VRC_1C_2.\frac{s + \left(\dfrac{C_1 + C_2}{RC_1C_2}\right)}{s^3LRC_1C_2 + s^2L(C_1 + C_2) + sRC_2 + 1} \qquad (19)$$

$$= \frac{VRC_1C_2}{LRC_1C_2}.\frac{s + \left(\dfrac{C_1 + C_2}{RC_1C_2}\right)}{s^3 + s^2\dfrac{L(C_1 + C_2)}{LRC_1C_2} + s\dfrac{RC_2}{LRC_1C_2} + \dfrac{1}{LRC_1C_2}} \qquad (20)$$

$$\boxed{i(s) = \frac{V}{L}.\frac{s + \left(\dfrac{C_1 + C_2}{RC_1C_2}\right)}{s^3 + s^2\dfrac{(C_1 + C_2)}{RC_1C_2} + s\dfrac{1}{LC_1} + \dfrac{1}{LRC_1C_2}}} \qquad (21)$$

3. Re-arrange terms as necessary.

Equation (21) is of the form:

$$\frac{s+n}{s^3+qs^2+rs+t} \qquad (22)$$

i.e. a cubic equation in s. I therefore need a Laplace transform of this form, into which (21) can be fitted and can then be transformed back into the time domain.

Looking at the table of transforms, the only one I can see which is of this form (once expanded) is number 16:

$$\frac{s+\alpha}{(s+a)(s+b)(s+c)} \qquad (23)$$

whose denominator, when expanded, becomes:

$$s^3+s^2(a+b+c)+s(ab+ac+bc)+abc \qquad (24)$$

and so equating coefficients we would have:

s^2 $$\frac{(C_1+C_2)}{RC_1C_2} \equiv (a+b+c) \qquad (25)$$

s $$\frac{1}{LC_1} \equiv (ab+ac+bc) \qquad (26)$$

constant (t) $$\frac{1}{LRC_1C_2} \equiv abc \qquad (27)$$

and also

$$\alpha \equiv \left(\frac{C_1+C_2}{RC_1C_2}\right) \qquad (28)$$

so in (25), (26) and (27) I have three equations and three unknowns, which should be solvable, although perhaps not easily. Before going any further, however, I have some concerns…

3a. Some concerns (and questions).

1. Is my general methodology correct? I have used the same method to find "i" for similar, simpler networks. Specifically, the series RLC and a series-parallel RLC (imagine C_2 replaced by a short circuit). In both cases, the resulting expressions enabled me to produce plots of "i" against time which were identical to those produced by PSpice simulations.

2. If the methodology is correct, have I made any errors – particularly in steps (1) to (21)? This is my third attempt, and on the two previous occasions, small algebraic slips crept in which invalidated everything which came afterwards.

3. Laplace transform 16 (step (23)) would result in the following transformed expression:

$$\frac{(\alpha - a)e^{-at}}{(b-a)(c-a)} + \frac{(\alpha - b)e^{-bt}}{(c-b)(a-b)} + \frac{(\alpha - c)e^{-ct}}{(a-c)(b-c)}$$

   which does not seem to contain any "oscillatory" term. I know from practical experience that the circuit will exhibit decaying sinusoidal oscillations, for certain values of R, L, C_1 and C_2, but I don't think this expression will be able to describe such behaviour [aside: when I analysed the series RLC and series-parallel RLC configurations, they fit transforms 24 and 28 respectively – both of which contain "oscillatory" sine terms].

Any advice, comments or pointers would be most welcome!