PHYS-2010: General Physics I
Course Lecture Notes
Section X

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Abstract

These class notes are designed for use of the instructor and students of the course PHYS-2010: General Physics I taught by Dr. Donald Luttermoser at East Tennessee State University. These notes make reference to the *College Physics, 7th Edition* (2005) textbook by Serway and Faughn.

X.   Rotational Equilibrium and Rotational Dynamics

A. Torque.

1.   The ability of a force to rotate a body about some axis is measured by a quantity called **torque** τ (Greek letter 'tau').

   a)   The torque due to a force of $\vec{F}$ has a magnitude of

$$\boxed{\tau = F\,d}\ . \qquad\qquad \text{(X-1)}$$

      i) $\tau \equiv$ torque (N·m).

      ii) $F \equiv$ applied force (N).

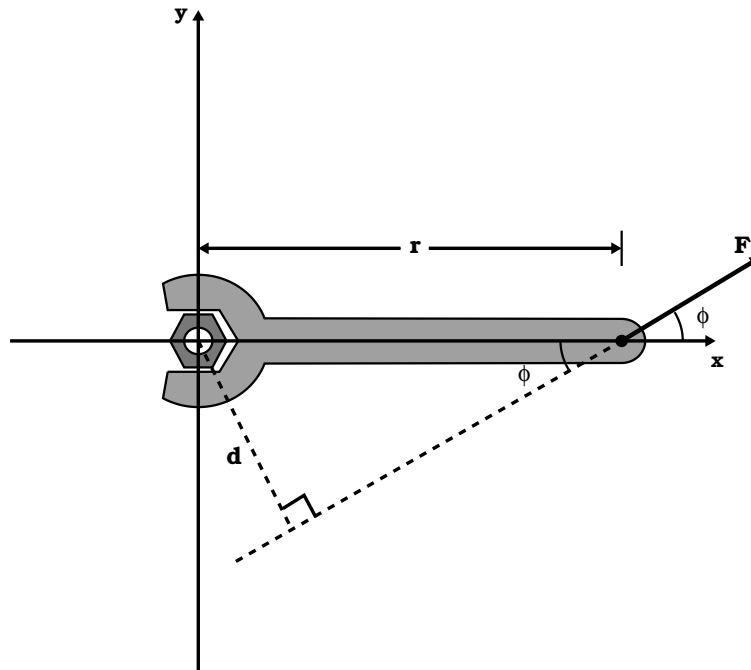   **iii)** $d \equiv$ lever arm (also called the *moment arm*) distance.

   b)   The **lever arm** is the $\perp$ distance from the axis of rotation to a line drawn along the direction of the force. Note that

$$d = r\,\sin\phi\ , \qquad\qquad \text{(X-2)}$$

where r is the magnitude of the displacement from the axis to the point of the applied force $\vec{F}$ and ϕ is the angle between the direction of $\vec{r}$ and the direction of $\vec{F}$.

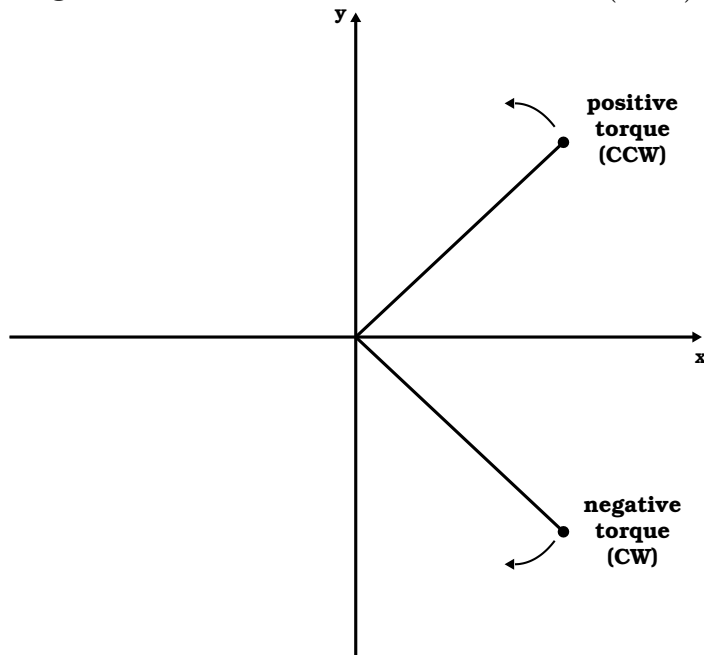   **c)**   As a result, we can rewrite the torque equation (Eq. X-1) as

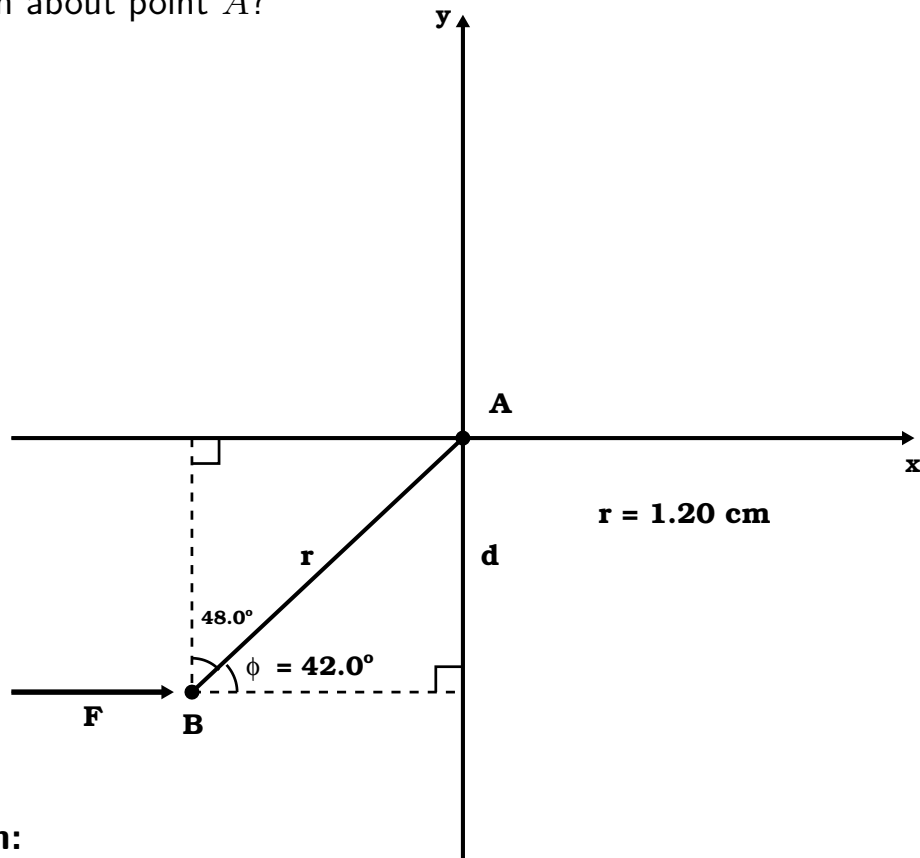$$\tau = r\,F\,\sin\phi\ . \qquad\qquad \text{(X-3)}$$

2. The **net torque** is found by summing all torques (*i.e.*, multiple forces acting on a rotating object).

$$\tau_{\text{net}} = \sum_{i=1}^{N} \tau_i = \tau_1 + \tau_2 + \cdots + \tau_N = F_1 d_1 + F_2 d_2 + \cdots + F_N d_N \ . \quad \text{(X-4)}$$

a) τ is positive if the rotation is counterclockwise (CCW).

b) τ is negative if the rotation is clockwise (CW).

Example X–1.   Problem 8.2 (Page 253) from the Serway & Faughn textbook: A steel band exerts a horizontal force of 80.0 N on a tooth at point B in Figure P8.2 (of the textbook). What is the torque on the root of the tooth about point A?

Solution:

The distance from point A to point B is $r = 1.20$ cm $= 1.20 \times 10^{-2}$ m. As can be seen from the figure above, the lever arm is

$$d = r \sin \phi = (1.20 \times 10^{-2} \text{ m}) \sin 42.0° = 8.03 \times 10^{-3} \text{ m} .$$

This gives a torque of

$$\tau = Fd = (80.0 \text{ N})(8.03 \times 10^{-3} \text{m}) = \boxed{0.642 \text{ N} \cdot \text{m} .}$$

Since the torque is positive, it is in the $\boxed{\text{counterclockwise}}$ direction.

B. Static Equilibrium.

1. The **first condition of equilibrium** is that the sum of all forces acting on an object is zero:

$$\sum_{i=1}^{N} F_i = 0 \qquad\qquad \text{(X-5)}$$

(see Eq. V-8).

2. The **second condition for equilibrium** asserts that if an object is in rotational equilibrium, the net torque acting on it about any axis must be zero:

$$\boxed{\sum_{i=1}^{N} \tau_i = 0} \;. \qquad\qquad \text{(X-6)}$$

3. A body in **static equilibrium** <u>must</u> satisfy 2 conditions:

    a) Resultant external force must be zero (Eq. X-5).

    b) Resultant external torque must be zero (Eq. X-6).

4. If a body is in equilibrium, the position of the axis of rotation is completely arbitrary when calculating net torque.

C. Center of Gravity.

1. The **center of gravity** of an object is the point where all of the weight of the object can be considered to be concentrated.
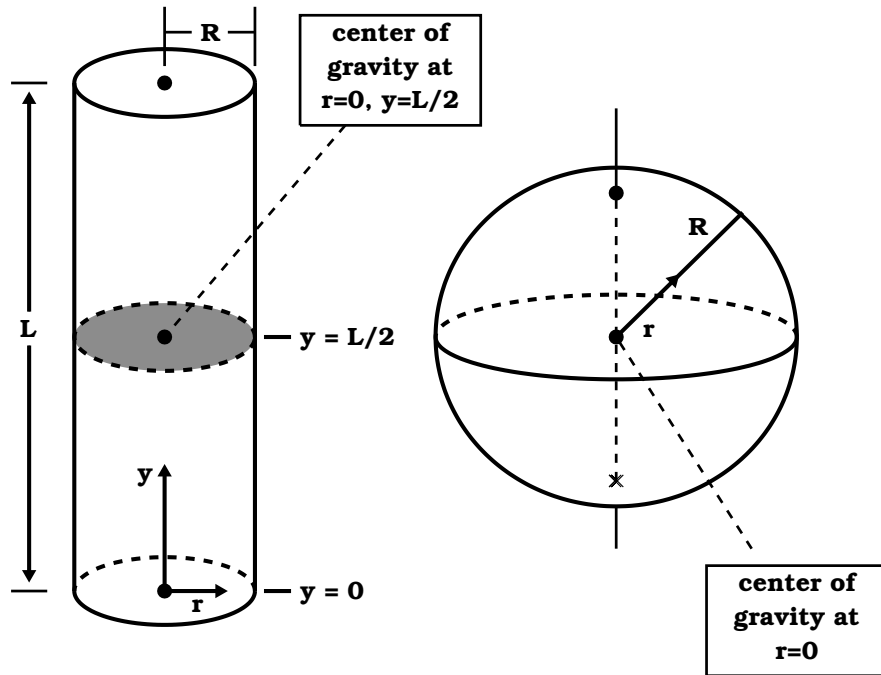
2. This also is called the **center of mass**:

$$x_{cg} = \frac{\sum_{i=1}^{N} m_i x_i}{\sum_{i=1}^{N} m_i} = \frac{m_1 x_1 + m_2 x_2 + \cdots + m_N x_N}{m_1 + m_2 + \cdots + m_N} \qquad \text{(X-7)}$$

$$y_{cg} = \frac{\sum_{i=1}^{N} m_i y_i}{\sum_{i=1}^{N} m_i} = \frac{m_1 y_1 + m_2 y_2 + \cdots + m_N y_N}{m_1 + m_2 + \cdots + m_N} \qquad \text{(X-8)}$$

$$z_{cg} = \frac{\sum_{i=1}^{N} m_i z_i}{\sum_{i=1}^{N} m_i} = \frac{m_1 z_1 + m_2 z_2 + \cdots + m_N z_N}{m_1 + m_2 + \cdots + m_N} \;. \qquad \text{(X-9)}$$

3. The center of gravity of a homogeneous, symmetric body must lie on the axis of symmetry.

D. Problem-Solving Strategy for Objects in Static Equilibrium.

1. Draw a diagram of the system.

2. Isolate the object that is being analyzed. Draw a free-body diagram showing all external forces acting on the system (do this for all masses in the system). Be careful to show the point where each force acts.

3. Chose coordinate axes and specify the positive sense of rotation. Then resolve the forces into components along the axes.

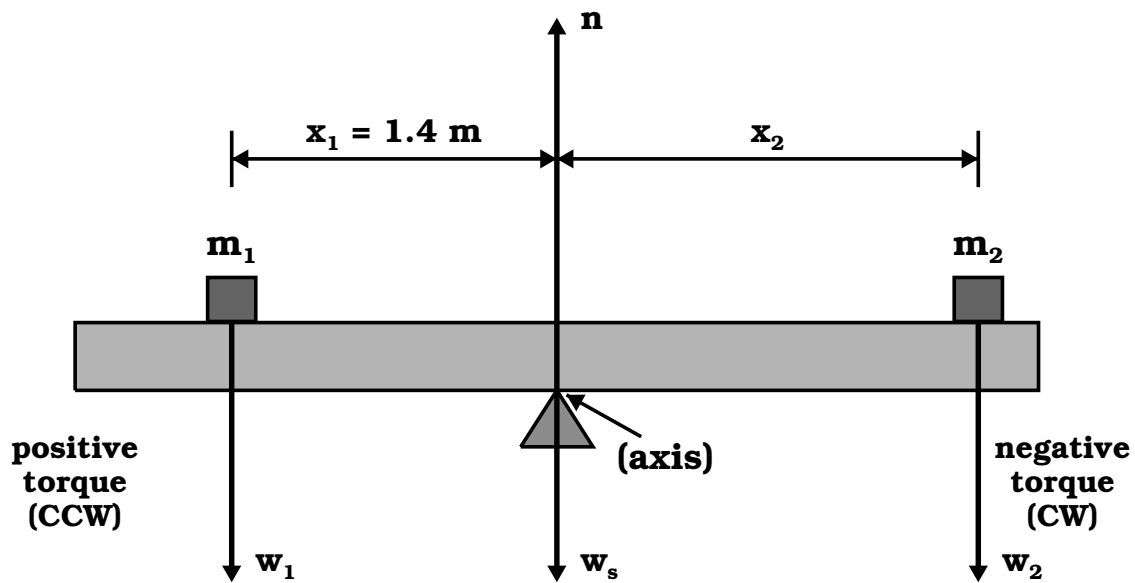**4.** Apply the first condition of equilibrium:

$$\sum F_x = 0 \quad \& \quad \sum F_y = 0 \ .$$

5. Chose an origin for calculating a net torque (usually an axis of rotation). Apply the second condition of equilibrium:

$$\sum \tau = 0 \; .$$

6. Solve the set of simultaneous equations from steps 4 and 5 and solve for the unknowns.

Example X–2.   Two children are playing on a balanced seesaw. One child with a mass of 42 kg sits 1.4 m from the center. Where on the other side must the second child sit if her mass is 34 kg?

Solution:

Since the seesaw is balanced, the sum of all torques must be zero. We will choose the axis at the point where the fulcrum is located. With this location of the axis, the torques from two of the forces, the weight of the seesaw w_s and the normal force on the fulcrum n, are zero since the lever arm is zero for both forces. Since the weight of mass 1, w_1, will cause the seesaw to rotate in the counterclockwise (CCW) direction, it will have a positive torque. Since the weight of mass 2, w_2, will cause the seesaw to rotate in the clockwise (CW) direction, it will have a negative torque.

Hence the torque equation becomes (remembering that $w = mg$)

$$\sum \tau = \tau_1 - \tau_2 + \tau_s + \tau_n = 0$$
$$= w_1 x_1 - w_2 x_2 + 0 + 0 = 0$$
$$= m_1 g x_1 - m_2 g x_2 = 0 .$$

Here, τ_1, τ_2, τ_s, and τ_n are the torques of mass 1, mass 2, the mass of the seesaw, and the normal force, respectively. Mass $m_1 = 42$ kg and $m_2 = 34$ kg. The lever arm on mass 1 is $x_1 = 1.4$ m. We want to find the lever arm on mass 2, which will be the distance that the second child is from the fulcrum (*i.e.*, the balance point), x_2, since this distance is perpendicular to the force vector w_2. Since the equation above, has only one unknown in it, we do not need the first condition of equilibrium for this problem (*e.g.*, $\sum F = n - w_1 - w_2 - w_s = 0$). Solving the above torque equation gives

$$x_2 = \frac{m_1}{m_2} x_1 = \frac{42 \text{ kg}}{34 \text{ kg}} (1.4 \text{ m}) = \boxed{1.7 \text{ m} .}$$

E. Torque and Moment of Inertia.

1. The tangential force on a rotating object is:

$$F_t = m a_t .$$

a) Multiplying this by the radius r of the rotation object gives

$$F_t r = m r a_t .$$

b) Since $a_t = r\alpha$ (see Eq. VIII-13), we can write

$$F_t r = m r^2 \alpha .$$

c) Finally, $F_t r$ is nothing more than the torque, so

$$\boxed{\tau = m r^2 \alpha} , \qquad\qquad \text{(X-10)}$$

where m is the mass of the rotating object, r is its radius, and α its angular acceleration.

d) The quantity $\underline{mr^2}$ is called the **moment of inertia** of mass $m \implies$ usually represented with an $\underline{I}$.

2. The total torque on a rotating object is then the sum of all the torques on each mass of the rotating object(s):

$$\tau_{\text{net}} = \sum_{i=1}^{N} \tau_i = \left(\sum_{i=1}^{N} m_i r_i^2 \right) \alpha , \qquad \text{(X-11)}$$

where the N is the number of the individual objects of mass m_i rotating about a common axis, each at a distance r_i from that axis.

a) The moment of inertia is then

$$\boxed{I = \sum_{i=1}^{N} m_i r_i^2} \qquad \text{(X-12)}$$

or

$$I = m_1 r_1^2 + m_2 r_2^2 + \cdots + m_N r_N^2$$

$$\implies [I] = \text{kg m}^2.$$

b) We can then rewrite Eq. (X-11) as

$$\boxed{\tau = I\alpha} . \qquad \text{(X-13)}$$

c) Note the similarity of this equation to Newton's 2nd Law of motion $\sum F = ma \implies$ Eq. (X-13) is the rotational counterpart to Newton's 2nd law.

d) **Force** & **mass** in linear motion correspond to **torque** & **moment of inertia**, respectively, in rotational motion.

3. Moments of inertia for common solids:

a) Solid sphere (with axis of rotation through center): $I = \frac{2}{5}MR^2$ ($M =$ mass, $R =$ radius of sphere).

b) Hoop or cylindrical shell (with axis of rotation through center, parallel to the wall): $I = MR^2$ ($R =$ outer radius).

c) Solid cylinder or disk (with axis of rotation through center, parallel to wall): $I = \frac{1}{2}MR^2$ ($R =$ radius of cylinder).

d) Long thin rod (rotation about center): $I = \frac{1}{12}ML^2$ ($L =$ length of rod).

e) Long thin rod (rotation about one end): $I = \frac{1}{3}ML^2$.

Example X–3.   Problem 8.35 (Page 258) from the Serway & Faughn textbook: A 150-kg merry-go-round in the shape of a uniform, solid, horizontal disk of radius 1.50 m is set in motion by wrapping a rope about the rim of the disk and pulling on the rope. What constant force must be exerted on the rope to bring the merry-go-round from rest to an angular speed of 0.500 rev/s in 2.00 s?

Solution:

This problem is clear enough without drawing a diagram. We first need to calculate the moment of inertia for the merry-go-round. Since we are told it has a solid disk shape rotating about its center,

$$I = \frac{1}{2}MR^2 = \frac{1}{2}(150 \text{ kg})(1.50 \text{ m})^2 = 169 \text{ kg} \cdot \text{m}^2 \ .$$

We also can calculate the angular acceleration based on the information given:

$$\alpha = \frac{\omega - \omega_\circ}{\Delta t} = \frac{0.500 \text{ rev/s} - 0}{2.00 \text{ s}} = 0.25 \text{ rev/s}^2 \cdot (2\pi \text{ rad/rev}) = \frac{\pi}{2} \text{ rad/s}^2 \ .$$

Using Eq. (X-1) in conjunction with Eq. (X-13), we have

$$\tau = Fd = I\alpha \ ,$$

where the lever arm here is just the radius of the merry-go-round: $d = r$. As such,

$$F = \frac{I\alpha}{r} = \frac{(169 \text{ kg} \cdot \text{m}^2)(\pi/2 \text{ rad/s}^2)}{1.50 \text{ m}} = \boxed{177 \text{ N}} \ .$$

F. Rotational Kinetic Energy.

1. A body rotating about some axis with an angular speed ω has a **rotational kinetic energy** of

    $$\boxed{\text{KE}_r = \frac{1}{2}I\,\omega^2} \ , \tag{X-14}$$

    which can be shown through the following argument.

    a) Total kinetic energy of all "mass elements" in a rotating body is

    $$\text{KE}_T = \sum_{i=1}^{N} \left(\frac{1}{2}m_i v_i^2\right) \ .$$

    b) However, $v = r\omega$ for an object in rotation, thus

    $$\text{KE}_T = \text{KE}_r = \sum_{i=1}^{N} \left(\frac{1}{2}m_i r_i^2 \omega_i^2\right) = \frac{1}{2}\omega^2 \sum_{i=1}^{N} \left(m_i r_i^2\right) \ ,$$

    where we have taken the angular speed out of the summation since all of the mass elements of the rotating body have the same angular speed.

    c) Finally, $I = \sum mr^2$, so

    $$\text{KE}_r = \frac{1}{2}I\omega^2 \ ,$$

    hence we have proven Eq. (X-14).

2. The conservation of mechanical energy now becomes

$$\boxed{(\mathrm{KE}_t + \mathrm{KE}_r + \mathrm{PE})_i = (\mathrm{KE}_t + \mathrm{KE}_r + \mathrm{PE})_f}\ .\qquad\text{(X-15)}$$

a) $\mathrm{KE}_t \equiv$ translational (straight line) KE.

b) $\mathrm{KE}_r \equiv$ rotational KE.

c) $\mathrm{PE} \equiv$ potential energy.

Example X–4.   Problem 8.41 (Page 258) from the Serway & Faughn textbook: A horizontal 800-N merry-go-round of radius 1.50 m is started from rest by a constant horizontal force of 50.0 N applied tangentially to the merry-go-round. Find the kinetic energy of the merry-go-round after 3.00 s. (Assume it is a solid cylinder.)

Solution:

This problem is clear enough without drawing a diagram. We first need to calculate the moment of inertia for the merry-go-round. Since we are told it is a solid cylinder rotating about its center,

$$I = \frac{1}{2}MR^2 = \frac{1}{2}\left(\frac{w}{g}\right)R^2 = \frac{1}{2}\left(\frac{800\ \mathrm{N}}{9.80\ \mathrm{m/s^2}}\right)(1.50\ \mathrm{m})^2 = 91.8\ \mathrm{kg\cdot m^2}\ .$$

We also can calculate the angular acceleration from the torque equations:

$$\alpha = \frac{\tau}{I} = \frac{F\cdot R}{I} = \frac{(50.0\ \mathrm{N})(1.50\ \mathrm{m})}{91.8\ \mathrm{kg\cdot m^2}} = 0.817\ \mathrm{rad/s^2}\ .$$

With this angular acceleration, we can now calculate the angular speed after 3.00 s starting from rest with

$$\omega = \omega_\circ + \alpha t = 0 + (0.817\ \mathrm{rad/s^2})(3.00\ \mathrm{s}) = 2.45\ \mathrm{rad/s}\ .$$

Finally, we can calculate the rotational kinetic energy with Eq. (X-14):

$$\mathrm{KE}_r = \frac{1}{2}I\omega^2 = \frac{1}{2}(91.8\ \mathrm{kg\cdot m^2})(2.45\ \mathrm{rad/s})^2 = \boxed{276\ \mathrm{J}}\ .$$

G. Angular Momentum.

1. We have the torque equation of Eq. (X-13):

$$\tau = I\alpha .$$

a) However, $\omega = \omega_{\circ} + \alpha t$ or $\omega = \omega_{\circ} + \alpha(\Delta t)$ if $t_{\circ} \neq 0$, thus

$$\alpha = \frac{\omega - \omega_{\circ}}{\Delta t} = \frac{\Delta \omega}{\Delta t} .$$

b) Then, we can rewrite torque as

$$\tau = I\left(\frac{\Delta \omega}{\Delta t}\right) = \frac{I\omega - I\omega_{\circ}}{\Delta t} .$$

2. The **angular momentum** of a rotating object is the moment of inertia multiplied by the angular velocity:

$$\boxed{L \equiv I\,\omega} . \qquad\qquad \text{(X-16)}$$

3. As a result, the torque also can be defined as the ratio of the change in angular momentum to the time interval Δt over which the change occurs:

$$\boxed{\tau = \frac{\text{change in ang. mom.}}{\text{time interval}} = \frac{L - L_{\circ}}{\Delta t} = \frac{\Delta L}{\Delta t}} . \qquad \text{(X-17)}$$

a) This is the rotational analogy to $F = \Delta p/\Delta t$ (see Eq. VII-2).

b) If $\sum \tau = 0$, then $\Delta L/\Delta t = 0$ and $L_{i} = L_{f}$ or $I_{i}\omega_{i} = I_{f}\omega_{f}$.

4. Conservation of Angular Momentum.

a) The angular momentum of a system is conserved when the net external torque acting on the system is zero:

$$\text{If } \sum \tau = 0 \ , \quad \text{then } L_{i} = L_{f} .$$

b) In an <u>isolated</u> (or closed) system, the energy, linear momentum, and angular momentum of the system remains constant.

Example X–5. (a) Calculate the angular momentum of Earth that arises from its spinning motion on its axis and (b) the angular momentum of Earth that arises from its orbital motion.

Solution (a):

This problem is clear enough without drawing a diagram. The angular momentum is $L = I\omega$. If we treat the Earth as a solid sphere spinning about its center, the moment of inertia is

$$I_{\rm rot} = \frac{2}{5}MR^2 = \frac{2}{5}M_\oplus R_\oplus^2 = \frac{2}{5}(5.98 \times 10^{24}~{\rm kg})(6.38 \times 10^6~{\rm m})^2$$
$$= 9.74 \times 10^{37}~{\rm kg~m}^2~.$$

The angular speed of the Earth's rotation is

$$\omega_{\rm rot} = \frac{2\pi}{T_{\rm rot}} = \frac{2\pi}{24~{\rm hr}} \cdot \frac{1~{\rm hr}}{3600~{\rm s}} = 7.27 \times 10^{-5}~{\rm rad/s}~.$$

This then gives the angular momentum of the Earth's spin as

$$L_{\rm rot} = I_{\rm rot}\,\omega_{\rm rot} = (9.74 \times 10^{37}~{\rm kg~m}^2)(7.27 \times 10^{-5}~{\rm rad/s})$$
$$= \boxed{7.08 \times 10^{33}~{\rm J \cdot s}}~.$$

Solution (b):

Considering the orbital motion, we consider the Earth to be a point mass a distance of $r_\oplus = 1.496 \times 10^{11}$ m from the Sun in a circular orbit, so

$$I_{\rm orb} = M_\oplus r_\oplus^2 = (5.98 \times 10^{24}~{\rm kg})(1.496 \times 10^{11}~{\rm m})^2 = 1.34 \times 10^{47}~{\rm kg~m}^2~.$$

The angular speed of the Earth in its orbit is

$$\omega_{\rm orb} = \frac{2\pi}{T_{\rm orb}} = \frac{2\pi}{1.00~{\rm yr}} \cdot \frac{1.00~{\rm yr}}{365.24~{\rm day}} \cdot \frac{1.00~{\rm day}}{24.00~{\rm hr}} \cdot \frac{1.00~{\rm hr}}{3600~{\rm s}}$$
$$= 1.99 \times 10^{-7}~{\rm rad/s}~.$$
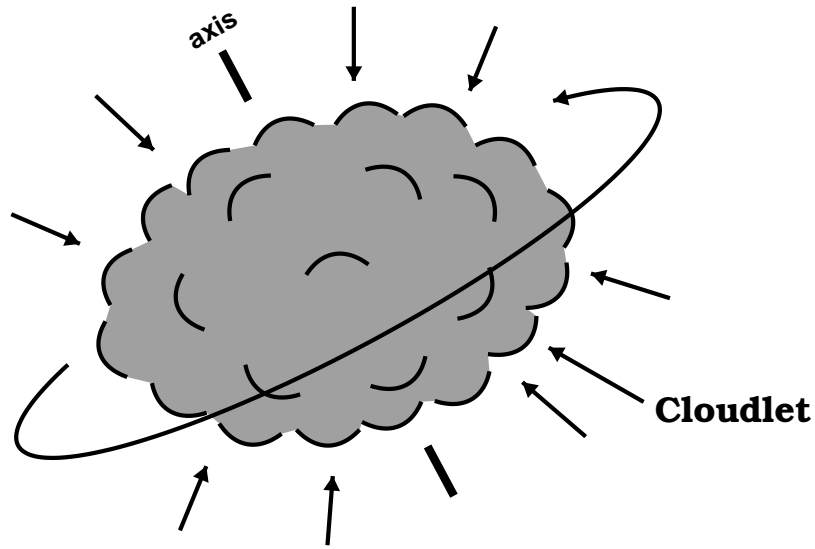
This then gives the angular momentum of the Earth's orbital motion as

$$L_{\text{orb}} = I_{\text{orb}}\,\omega_{\text{orb}} = (1.34 \times 10^{47} \text{ kg m}^2)(1.99 \times 10^{-7} \text{ rad/s})$$

$$= \boxed{2.66 \times 10^{40} \text{ J} \cdot \text{s}}.$$

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5. The Formation of the Solar System.

a) The solar system started out as a giant (more or less spherical) gas cloud a few light years in diameter and rotating very slowly (a few meters per second on the outer rim) $\Longrightarrow$ we currently see such clouds in star forming regions of our Milky Way Galaxy.

b) An instability occurred in the cloud from either the passage of one of the spiral arms of the Milky Way or a supernova shock wave passing through the cloud (aluminum isotope ratios in the solar system suggest the latter of these two causes likely happened). This set up a compression in the cloud which caused $F_g > (P \cdot A)$ (the weight of the cloud exceeded the internal pressure that was keeping it stable) $\Longrightarrow$ the cloud began to collapse.

  i) Initially these clouds, called **Giant Molecular Clouds** (GMC), have thousands of solar mass worth of material in them.

  ii) Within the GMC, turbulence is common just as in a thunderstorm here on Earth. This turbulence causes localized spinning "eddies" to exist within the GMC.

iii) During the collapse, gravitational instabilities cause the GMC to fragment into smaller **cloudlets** that range in mass from approximately 0.1 $M_\odot$ up to approximately 100 $M_\odot$ in mass each.

iv) The eddies from the GMC induce an initial spin to each cloudlet as it begins to collapse (see the diagram below).

v) The more massive the cloudlet, the faster it collapses as shown in the following table.

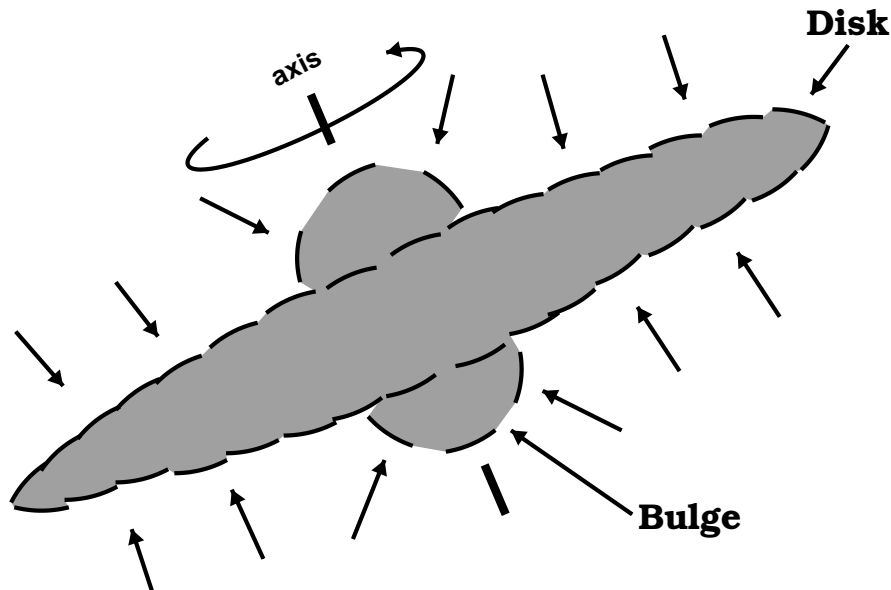| Cloudlet Mass ($M_\odot$)	Contraction Time (Years)
0.1	10^8
1.0	10^7
3.0	10^6
10	10^5
100	10^4

c)   As the solar cloudlet (often referred to as the **solar neb-ula**) collapsed, it began to spin faster due to the conservation of angular momentum:

$$L_i = L_f$$
$$I_i\omega_i = I_f\omega_f$$
$$\frac{2}{5}M_i r_i^2 \omega_i = \frac{2}{5}M_f r_f^2 \omega_f \ ,$$

and since the mass is conserved during collapse, $M_i = M_f$,

$$r_i^2 \omega_i = r_f^2 \omega_f$$
$$\omega_f = \left(\frac{r_i}{r_f}\right)^2 \omega_i \ .$$

d)   The faster the cloud spins, the flatter it got $\Longrightarrow$ "centrifugal" acceleration as shown in the following diagram.

i)   A central bulge formed (called **protostars**, and in the case of the Sun, the **protosun**).

ii)   A disk formed surrounding the central bulge.

e) P and T got so high in the central bulge that thermonuclear reactions began $\Longrightarrow$ **the Sun was born**.

f) Meanwhile in the cooler disk, the planets (initially called **protoplanets**) formed through the process of condensation (solids forming from gas), advection (bigger particles forming from smaller ones), and accretion (large objects forming even larger objects through collisions).