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A new interpretation of the NUT metric in general relativity

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Abstract. The NUT metric is interpreted as the field of (*a*) a mass around the origin of coordinates, and (*b*) a semi-infinite massless source of angular momentum.

1. *Introduction*. This paper is about the metric discovered by Newman, Unti and Tamburino (6), hereafter abbreviated to NUT. This metric is a solution of Einstein's empty space equations

$$R_{ik} = 0, \tag{1·1}$$

is stationary, axially symmetric and tends to flatness (except in one direction) as the radial coordinate tends to infinity. It reduces to the Schwarzschild metric if a certain arbitrary constant is put zero. Moreover, as I intend to show in a later paper, the metric is a member of a class of stationary solutions of (1·1) discovered many years ago by Papapetrou (7).

These simple properties challenge relativists to give a physical interpretation of the metric. An ingenious one has been offered by Misner (4, 5). Misner introduces a periodic time coordinate, and every observer at rest in the coordinate system moves on a closed time-like world-line. Such a space-time, in which an observer moves forward in time only to find himself in his own past, resembles somewhat the world of Dr Who (9), and is physically puzzling, to put it midly.

In this paper I attempt a different interpretation. Briefly, I claim that the NUT metric describes the field of a spherically symmetric mass together with a semi-infinite massless source of angular momentum along the axis of symmetry. I justify this claim mainly by an appeal to linearized general relativity.

In section 2 I introduce the NUT metric and describe some of its properties. The main original work of the paper is the analysis of sources of the linearized NUT metric in section 3. This is followed in section 4 with a precise mathematical specification of the manifold which I associate with the NUT metric. The paper concludes with an attempt, in section 5, to reconcile my interpretation with previous work on the group of motions admitted by the NUT metric.

2. *The NUT metric*. The NUT metric is

$$ds^2 = U(dt + 4a\sin^2\tfrac{1}{2}\theta\, d\phi)^2 - U^{-1}dr^2 - (r^2+a^2)(d\theta^2 + \sin^2\theta\, d\phi^2), \tag{2·1}$$

where

$$U = 1 - \frac{2(mr+a^2)}{r^2+a^2}, \tag{2·2}$$

m and a being constants ($m > 0$). The domain of the coordinates corresponding to my interpretation of (2·1) will be given in section 4. The following are some obvious properties of the metric.

I. If $a = 0$ the solution reduces to the Schwarzschild metric for an isolated spherically symmetric mass m.

II. Singularities arise because U vanishes when $r = m \pm \sqrt{(m^2+a^2)}$. If, as I shall assume, r must be non-negative (cf. Misner (4)), only the positive sign is relevant. As r decreases below $m + \sqrt{(m^2+a^2)}$ it becomes a time-like coordinate, as in the Schwarzschild solution. (However, unlike the Schwarzschild solution, (2·1) has no singularity at $r = 0$.)

III. The metric (2·1) is singular along the axis of symmetry $\theta = 0$ and $\theta = \pi$, because the signature is not $+ - - -$. The singularity along $\theta = 0$ can be removed by a transformation to cartesian coordinates (as shown in the Appendix), but this transformation does not remove the singularity at $\theta = \pi$. This does not mean, of course, that the latter cannot be removed by some other transformation. *Nevertheless, I shall tentatively assume that $\theta = \pi$ is a physical singularity representing a source of the field.* The justification for this assumption will be given in section 3.

In view of I and II *I shall suppose that m represents the mass of a spherically symmetric body centre at the origin* $r = 0$. The zero and change of sign of U will not be considered further, and the region where these occur will be excluded from the solution, as explained in section 4.

The main purpose of the paper is to propose an interpretation for the constant a, which measures the strength of the physical singularity on $\theta = \pi$. It is in order to eliminate this singularity that Misner (4) is led to introduce a periodic time coordinate.

Misner states that the orthonormal components of the Riemann tensor are functions of r only, and are non-singular for all r. This does not of course prove that the manifold is free from singularities: to show this it is only necessary to mention that near the apex of a cone in E^3 the Riemann tensor is perfectly well behaved—it vanishes.

3. *Physical interpretation.* In this section I shall investigate the meaning of the singularity at $\theta = \pi$ in the metric (2·1). From the presence of a term in $d\phi\, dt$ I conclude that the metric refers to rotating bodies. The asymptotic form of metric for a rotating sphere of mass m and angular momentum h at the origin is known from the linear approximation to be

$$ds^2 = -\left(1+\frac{2m}{r}\right)dr^2 - r^2(d\theta^2+\sin^2\theta\, d\phi^2) + \left(1-\frac{2m}{r}\right)dt^2 + \frac{2h}{r}\sin^2\theta\, d\phi\, dt. \quad (3·1)$$

The asymptotic form of (2·1) is

$$ds^2 = -\left(1+\frac{2m}{r}\right)dr^2 - r^2(d\theta^2+\sin^2\theta\, d\phi^2) + \left(1-\frac{2m}{r}\right)dt^2 + 4a\left(1-\frac{2m}{r}\right)(1-\cos\theta)\, d\phi\, dt. \quad (3·2)$$

The term in $d\phi\, dt$ is larger in (3·2) than in (3·1) and so we may assume that the source of rotation is something 'stronger' than a rotating sphere. An obvious possibility is a rotating source along the singularity $\theta = \pi$. If this source had uniform line-mass density σ, a term of order r^σ in g_{tt} would be expected. If the line-mass density were dependent on position and decreased as $r \to \infty$ in some suitable way its effect on g_{tt} would be less drastic, but, nevertheless, some extra multipole terms in g_{tt} would be expected. I conclude that there is no mass source distributed along $\theta = \pi$.

I shall therefore tentatively investigate the hypothesis that the term in $d\phi\, dt$ arises from a linear source of pure angular momentum along $\theta = \pi$, that is from a massless rotating rod. To do this I put $m = 0$ and consider the approximation of (2·1) linear in the parameter a, namely

$$ds^2 = -dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) + dt^2 + 4a(1-\cos\theta)\, d\phi\, dt. \tag{3·3}$$

This must satisfy the equations of linearized relativity theory, in which the sources can be fairly easily identified. I shall therefore examine (3·3) as a solution of the linearized theory, using some recent work on multipole solutions of that theory (1).

Define the Minkowski η_{ik}, η^{ik} as usual by

$$\eta_{ik} = e_i \delta_{ik}, \quad \eta^{ik} = e_i \delta^{ik}, \tag{3·4}$$

where $e_i = -1$ if $i = 1, 2, 3$ and $e_4 = 1$. Write for the metric tensor

$$g_{ik} = \eta_{ik} + \gamma_{ik}, \tag{3·5}$$

where the γ_{ik} are small, and products of γ_{ik} and their derivatives may be neglected. Introduce the auxiliary quantities

$$\gamma^*_{ik} = \gamma_{ik} - \tfrac{1}{2}\eta_{ik}\eta^{ab}\gamma_{ab}, \tag{3·6}$$

and use the harmonic coordinate condition

$$\eta^{ab}\gamma^*_{ia,b} = 0, \tag{3·7}$$

where a comma denotes partial differentiation. The field equations

$$R_{ik} - \tfrac{1}{2}g_{ik}R = -8\pi T_{ik} \tag{3·8}$$

now become, in the linear approximation (3)

$$\eta^{ab}\gamma^*_{ik,ab} = -16\pi T_{ik}. \tag{3·9}$$

In the stationary case this reduces to

$$\eta^{\alpha\beta}\gamma^*_{ik,\alpha\beta} = -16\pi T_{ik} \quad (\alpha, \beta = 1, 2, 3). \tag{3·10}$$

Consider a rotating source, and let ξ_α be the position vector of a point Q in the body relative to an origin fixed in some Lorentz frame. Introduce the following quantities

$$\left.\begin{aligned} M &= \int_V T_{44}\, dv, & M_\mu &= \int_V T_{44}\xi_\mu\, dv, & M_{\mu\nu} &= \int_V T_{44}\xi_\mu\xi_\nu\, dv, \\ A_\alpha &= \int_V T_{4\alpha}\, dv, & A_{\alpha|\beta} &= \int_V T_{4\alpha}\xi_\beta\, dv, & S'_{\alpha\beta} &= \int_V T_{\alpha\beta}\, dv, \end{aligned}\right\} \tag{3·11}$$

where V is a fixed volume enclosing the source. The linearized identities are (3)

$$\frac{\partial T_{i\alpha}}{\partial \xi_\alpha} - \frac{\partial T_{i4}}{\partial x_4} = 0 \quad (i = 1, 2, 3, 4), \tag{3·12}$$

where T_{ik} is a function of the coordinates of the source-point Q, and in general of x_4. These identities impose conditions on the moments (3·11). We are interested in the stationary case and suppose the coordinate frame chosen so that there is no dependence on x_4. Following my previous procedure one can show that the conditions on the moments defined in (3·11) are

$$\left.\begin{aligned} M &= \text{const.}, \quad & M_\mu &= 0, \quad & M_{\mu\nu} &= \text{const.}, \\ A_\alpha &= 0, \quad & A_{\alpha|\beta} &= -A_{\beta|\alpha} = k_{\alpha\beta}(\text{const.}), \quad & S_{\alpha\beta} &= 0. \end{aligned}\right\} \tag{3·13}$$

There are no surprises here: the formulae state that the total mass M is constant; that the linear momentum and first mass moments vanish (the origin having been chosen at the centre of mass); that the angular momentum and the second mass moments are constant. The only result not immediately obvious is that $S_{\alpha\beta}$ vanishes. It is to be emphasized that all this refers to the linear approximation only.

Our aim is to obtain the γ_{ik} corresponding to a semi-infinite source of angular momentum. The first step is to integrate (3·10) for the γ^*_{ik}, and the result in the stationary case up to the second order moments is (1)

$$\left.\begin{aligned} \gamma^*_{\alpha\beta} &= 0, \\ \gamma^*_{\alpha 4} &= -\frac{4x_\mu A_{\alpha|\mu}}{R^3}, \\ \gamma^*_{44} &= -\frac{4M}{R} - \frac{2x_\mu x_\nu}{R^5}(3M_{\mu\nu} - \delta_{\mu\nu}\delta_{\alpha\beta}M_{\alpha\beta}), \end{aligned}\right\} \tag{3·14}$$

where $R^2 = \delta_{\alpha\beta}x_\alpha x_\beta$, x_α being the field point, and where (3·13) has been used. (3·14) agrees with the results of Sachs and Bergmann (8). We now suppose that the only source is one of angular momentum $2a$ units per unit length uniformly distributed along the negative z-axis, so that the mass moments M and $M_{\mu\nu}$ vanish. Writing $x_1 = x$, $x_2 = y$, $x_3 = z$, we have for unit length of source

$$-A_{2|1} = A_{1|2} = a. \tag{3·15}$$

On integrating the contributions from source elements along the negative z-axis we obtain

$$-y^{-1}\gamma^*_{14} = x^{-1}\gamma^*_{24} = 4a\int_{-\infty}^{0} \frac{d\zeta}{[x^2+y^2+(z-\zeta)^2]^{\frac{3}{2}}} = \frac{4a}{x^2+y^2}\left(1 - \frac{z}{R}\right), \tag{3·16}$$

the remaining γ^*_{ik} being zero. Using (3·6) we can write down the perturbed Minkowski metric, and transforming to spherical polar coordinates from cartesians (putting $R = r$) we obtain (3·3). Thus *a source of angular momentum uniformly distributed along the negative z-axis yields, in the linearized theory, the asymptotic metric* (3·3). I conclude that the constant a in the NUT solution refers to such a source.

4. *Precise specification of the solution.* I shall now specify in detail the NUT solution, as it is interpreted in this paper.

With the metric (2·1) I associate a differentiable manifold. Singular regions of the metric will be omitted from the manifold. Indeed not only will singular regions be excluded but also a region containing closed time-like and null lines, namely the region where $g_{\phi\phi} \geqslant 0$. From (2·1) this is seen to be where

$$\pi \geqslant \theta \geqslant 2\tan^{-1}\sqrt{\left(\frac{r^2+a^2}{4a^2U}\right)} > 0.$$

The manifold discussed in this paper therefore consists of metric (2·1) together with the following domains of coordinates

$$\left.\begin{aligned} &r > r_0 > m+\sqrt{(m^2+a^2)}, \quad 0 < \theta < 2\tan^{-1}\sqrt{\left(\frac{r^2+a^2}{4a^2U}\right)} < \pi, \\ &0 \leqslant \phi \leqslant 2\pi, \quad -\infty < t < \infty, \end{aligned}\right\} \tag{4·1}$$

r_0 being a constant. The points

$$(r,\theta,0,t) \quad \text{and} \quad (r,\theta,2\pi,t) \tag{4·2}$$

are to be identified. The space-like line $\theta = 0$ can be included in the manifold by using a cartesian coordinate patch (Appendix). There are holes in the manifold at

$$r \leqslant r_0, \quad \theta \geqslant 2\tan^{-1}\sqrt{\left(\frac{r^2+a^2}{4a^2U}\right)},$$

and the manifold is incomplete in the sense that not all geodesics can be extended in both directions to arbitrarily large values of their affine parameters (4, 2). To complete the manifold one would need to fill the holes in by a solution of Einstein's interior equations (3·8), though this will not be attempted here.

5. *Groups of motions of NUT space-time. NUT* space-time admits a four parameter group of motions of which the Killing vectors are (Misner (4))

$$\left.\begin{aligned} \xi_b &= \partial/\partial t, \\ \xi_x &= -\sin\phi\frac{\partial}{\partial\theta}-\cos\phi\left(\cot\theta\frac{\partial}{\partial\phi}+2a\tan\tfrac{1}{2}\theta\frac{\partial}{\partial t}\right), \\ \xi_y &= \cos\phi\frac{\partial}{\partial\theta}-\sin\phi\left(\cot\theta\frac{\partial}{\partial\phi}+2a\tan\tfrac{1}{2}\theta\frac{\partial}{\partial t}\right), \\ \xi_z &= \frac{\partial}{\partial\phi}-2a\frac{\partial}{\partial t}. \end{aligned}\right\} \tag{5·1}$$

The commutation relations are the same as in the Schwarzschild case, but the actions of the groups on the two spaces are quite different. In particular, whereas the orbits of a point P under ξ_x, ξ_y, ξ_z form in the Schwarzschild space-time a 2-sphere, $r =$ const., $t =$ const., in NUT space-time these orbits are three dimensional.

Since r does not appear in (5·1) the hypersurfaces $r =$ const. in the NUT space-time are homogeneous according to the group theoretical point of view. The question arises how this is to be reconciled with the interpretation of this paper. The latter definitely requires that $\theta = \pi$ corresponds to a physical singularity, but this may not be in contradiction to the previous paragraph because ξ_x and ξ_y are also singular at

$\theta = \pi$. These singularities were not analysed in Misner's paper, and may turn out to have physical significance. The situation may be analogous to a spherical surface in E^3 with the south pole removed: this admits three local solutions of Killing's equations (corresponding to ξ_x, ξ_y, ξ_z with $a = 0$) but contains a singularity at an assigned point. This raises the question of the global significance of solutions of Killing's equations if the manifold is not complete.

We still have to explain the homogeneity on

$$r = \text{const.}, \quad 0 \leqslant \theta < \pi, \quad 0 \leqslant \phi \leqslant 2\pi, \quad -\infty < t < \infty.$$

Physically this is surprising, but there are some analogies in Newtonian gravitation theory which help to understand it. First, we recall that the gravitational field of an infinite plane is homogeneous on either side of it. Naïvely one might expect the field to be stronger as one approaches the plane, but the theory says this does not happen. This example shows that an infinite source can give rise to a homogeneous field in a way surprising to the uninstructed. A more striking example is the following. Consider a semi-infinite line mass on $-\infty < z \leqslant 0$. One finds from ordinary attraction theory that *the z-component of the attraction at a point P depends only on the distance OP from the origin* (though the x- and y-components depend on the perpendicular distance of P from the rod). In this example the z-component is homogeneous on any sphere centre O, in spite of the source being a semi-infinite rod.

Unfortunately we do not have any Newtonian theory of the attraction of a rotating source, so we cannot check the homogeneity in the Newtonian version. But these analogies suggest that the interpretation proposed here need not conflict with the group-theoretical results.

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APPENDIX

Removal of the singularity on $\theta = 0$

The transformation

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta, \quad t = t, \tag{A. 1}$$

takes the NUT metric into

$$ds^2 = U\left[dt + \frac{2a(x\,dy - y\,dx)}{r(r+z)}\right]^2 - U^{-1}dr^2$$
$$- (r^2+a^2)r^{-4}\left[(x\,dz - z\,dx)^2 + (y\,dz - z\,dy)^2 + (x\,dy - y\,dx)^2\right], \tag{A. 2}$$

where $r = +(x^2+y^2+z^2)^{\frac{1}{2}}$. The transformation (A. 1) is not regular on $\theta = 0$, $\theta = \pi$ since the Jacobian vanishes there; nevertheless one may work with (A. 2) instead of (2·1) because it satisfies the field equations (1·1) except where it is not of class C^2. Evidently (A. 2) is non-singular on $x = 0$, $y = 0$, $z > 0$ (corresponding to $\theta = 0$) and singular on $x = 0$, $y = 0$, $z < 0$ (corresponding to $\theta = \pi$).

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