

From the point of view of the angular-momentum commutation relations alone, it might not appear obvious why l cannot be a half-integer. It turns out that several arguments can be advanced against half-integer l -values. First, for half-integer l , and hence for half-integer m , the wave function would acquire a minus sign,

$$e^{im(2\pi)} = -1, \quad (3.6.40)$$

under a 2π rotation. As a result, the wave function would not be single-valued; we pointed out in Section 2.4 that the wave function must be single-valued because of the requirement that the expansion of a state ket in terms of position eigenkets be unique. We can prove that if \mathbf{L} , defined to be $\mathbf{x} \times \mathbf{p}$, is to be identified as the generator of rotation, then the wave function must acquire a plus sign under a 2π rotation. This follows from the fact that the wave function for a 2π -rotated state is the original wave function itself with no sign change:

$$\begin{aligned} \langle \mathbf{x}' | \exp\left(\frac{-iL_z 2\pi}{\hbar}\right) | \alpha \rangle &= \langle x' \cos 2\pi + y' \sin 2\pi, y' \cos 2\pi - x' \sin 2\pi, z' | \alpha \rangle \\ &= \langle \mathbf{x}' | \alpha \rangle, \end{aligned} \quad (3.6.41)$$

where we have used the finite-angle version of (3.6.6). Next, let us suppose $Y_l^m(\theta, \phi)$ with a half-integer l were possible. To be specific, we choose the simplest case, $l = m = \frac{1}{2}$. According to (3.6.34) we would have

$$Y_{1/2}^{1/2}(\theta, \phi) = c_{1/2} e^{i\phi/2} \sqrt{\sin \theta}. \quad (3.6.42)$$

From the property of L_- [see (3.6.36)] we would then obtain

$$\begin{aligned} Y_{1/2}^{-1/2}(\theta, \phi) &= e^{-i\phi} \left(-\frac{\partial}{\partial \theta} + i \cot \theta \frac{\partial}{\partial \phi} \right) (c_{1/2} e^{i\phi/2} \sqrt{\sin \theta}) \\ &= -c_{1/2} e^{-i\phi/2} \cot \theta \sqrt{\sin \theta}. \end{aligned} \quad (3.6.43)$$

This expression is not permissible because it is singular at $\theta = 0, \pi$. What is worse, from the partial differential equation

$$\begin{aligned} \left\langle \hat{n} | L_- | \frac{1}{2}, -\frac{1}{2} \right\rangle &= -i\hbar e^{-i\phi} \left(-i \frac{\partial}{\partial \theta} - \cot \theta \frac{\partial}{\partial \phi} \right) \left\langle \hat{n} | \frac{1}{2}, -\frac{1}{2} \right\rangle \\ &= 0 \end{aligned} \quad (3.6.44)$$