Solutions to Assignment # 1 (PHYS 1414)

1.

(a) Let x be the displacement of the wedge

$\therefore$ displacement of the particle relative to ground

$$= \frac{h}{\tan\phi} - x$$

Note that the center of mass of the system does not move.

$$\therefore \quad m\left(\frac{h}{\tan\phi} - x\right) - Mx = 0$$

$$\frac{mh}{\tan\phi} - (m+M)x = 0$$

$$x = \frac{mh}{(m+M)\tan\phi}$$

$$x = \frac{mh\cot\phi}{m+M}$$

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(b)

horiz. speed of the particle relative to the ground

$= u\cos\phi - v$

u: speed of particle relative to wedge

v: speed of wedge relative to ground

By the principle of conservation of momentum,

$$m(u\cos\phi - v) - Mv = 0$$

$$mu\cos\phi - (m+M)v = 0$$

$$v = \frac{mu\cos\phi}{m+M} \quad \text{——} (*)$$

By the conservation of mechanical energy,

$$\frac{1}{2}m(u\cos\phi - v)^2 + \frac{1}{2}m(u\sin\phi)^2 + \frac{1}{2}Mv^2 = mgh$$

$$\frac{1}{2}m(u^2\cos^2\phi - 2uv\cos\phi + v^2) + \frac{1}{2}mu^2\sin^2\phi + \frac{1}{2}Mv^2 = mgh$$

$$\frac{1}{2}mu^2 - muv\cos\phi + \frac{1}{2}mv^2 + \frac{1}{2}Mv^2 = mgh$$

From $*$ $\quad u = \dfrac{(m+M)v}{m\cos\phi}$

Hence,

$$\frac{1}{2} m \frac{(m+M)^2 v^2}{m^2 \cos^2\phi} - m\left(\frac{(m+M)v}{m\cos\phi}\right) v\cos\phi + \frac{1}{2} m v^2 + \frac{1}{2} M v^2 = mgh$$

$$\frac{1}{2} \frac{(m+M)^2}{m\cos^2\phi} v^2 - (m+M)v^2 + \frac{1}{2} m v^2 + \frac{1}{2} M v^2 = mgh$$

$$v^2\left[\frac{1}{2}\frac{(m+M)^2}{m\cos^2\phi} - \frac{(m+M)}{2}\right] = mgh$$

$$\frac{v^2}{2}(m+M)\left[\frac{m+M}{m\cos^2\phi} - 1\right] = mgh$$

$$\frac{v^2}{2}(m+M)\left[\frac{m+M-m\cos^2\phi}{m\cos^2\phi}\right] = mgh$$

$$v^2 = \frac{2m^2gh\cos^2\phi}{(M+m\sin^2\phi)(m+M)}$$

$$v = \sqrt{\frac{2gm^2h\cos^2\phi}{(M+m\sin^2\phi)(m+M)}} \quad \#$$

(c)

From (a), (b) and

$$\begin{cases} v^2 = 0 + 2ax \\ v = 0 + at \end{cases} \Rightarrow \begin{cases} v^2 = 2ax \\ v = at \end{cases} \Rightarrow t = \frac{2X}{v}$$

where a: accel. of wedge.

$$\therefore \quad t = \frac{2mh\cot\phi}{m+M}\sqrt{\frac{(M+m\sin^2\phi)(m+M)}{2gm^2h\cos^2\phi}}$$

$$t = \sqrt{\frac{2h(M+m\sin^2\phi)}{(m+M)g\sin^2\phi}}$$

(d)

The same answer with (a).

Because the frictional forces are internal forces of the system (particle + wedge).

2.

(a)

$$-Mg = M\frac{dv}{dt} - (-U_0)\frac{dM}{dt}$$ $\uparrow$ +ve

$$-Mg = M\frac{dv}{dt} + U_0\frac{dM}{dt}$$

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(b) $$-M\left(\frac{dv}{dt} + g\right) = U_0\frac{dM}{dt}$$

$$-\int_0^t \left(\frac{dv}{dt} + g\right) dt = U_0 \int_{M_0+m_0}^{M_0} \frac{dM}{M}$$

$$-\int_0^v dv - gt = U_0 \ln\left(\frac{M_0}{M_0+m_0}\right)$$

$$-v - gt = U_0 \ln\left(\frac{M_0}{M_0+m_0}\right)$$

$$v = U_0 \ln\left(\frac{M_0+m_0}{M_0}\right) - gt$$

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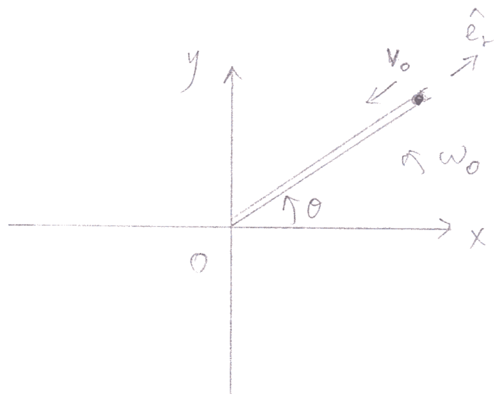
3.

(a) $\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (r\ddot{\theta} - 2\dot{r}\dot{\theta})\hat{e}_\theta$



Force along the rod $= 0$

$\dot{\theta} = \omega_0 =$ constant

$\therefore \quad \ddot{r} - r\omega_0^2 = 0$

$$\dot{r}\frac{d\dot{r}}{dr} - r\omega_0^2 = 0$$

$$\int_{-v_0}^{v} \dot{r}\,d\dot{r} = \omega_0^2 \int_{l}^{r} r\,dr$$

$$\frac{\dot{r}^2}{2}\Bigg|_{-v_0}^{v} = \omega_0^2 \frac{r^2}{2}\Bigg|_{l}^{r}$$

$$v^2 - v_0^2 = \omega_0^2(r^2 - l^2)$$

$$v^2 = \omega_0^2(r^2 - l^2) + v_0^2$$

Use $v_0 = l\omega_0$

$$v^2 = \omega_0^2(r^2 - l^2) + l^2\omega_0^2$$

$$v^2 = \omega_0^2 r^2$$

$$v = -\omega_0 r \quad \#$$

(v is opposite to the measurement of r.)

(b)

$$v = \frac{dr}{dt} = -w_0 r$$

$$\Rightarrow \int_{l}^{\frac{l}{2}} \frac{dr}{r} = -w_0 \int_0^t dt$$

$$\ln r \Big|_{l}^{\frac{l}{2}} = -\omega_0 t$$

$$\ln \frac{1}{2} = -\omega_0 t$$

$$t = \frac{1}{\omega_0} \ln 2$$

#

(c)

In fact, $v = \frac{dr}{dt} = -w_0 r$

$$\Rightarrow \int_{l}^{r} \frac{dr}{r} = -w_0 \int_0^t dt$$

$$\ln \frac{r}{l} = -w_0 t$$

$$\frac{r}{l} = e^{-w_0 t}$$

$$r = l e^{-w_0 t}$$

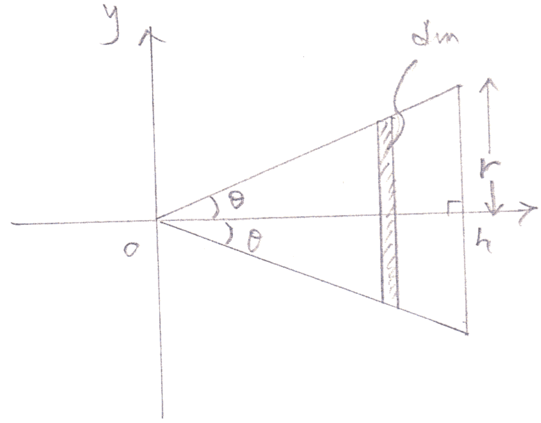
i.e. $r \to 0$ when $t \to \infty$

$\therefore$ The particle can never reach O.

4.

(a)

$$\bar{x} = \frac{\int x\,dm}{\int dm} \quad \text{where} \quad dm = \pi (x \tan\theta)^2 \, dx \, \rho$$



$$\bar{x} = \frac{\int_0^h \rho \pi \tan^2\theta \; x^3 \, dx}{\int_0^h \rho \pi \tan^2\theta \; x^2 \, dx}$$

$$\bar{x} = \frac{\int_0^h x^3 \, dx}{\int_0^h x^2 \, dx}$$

$$\begin{cases} \tan\theta = \dfrac{r}{h} = \text{constant} \\ \rho = \text{volume density} \end{cases}$$

$$\bar{x} = \frac{h^4/4}{h^3/3}$$

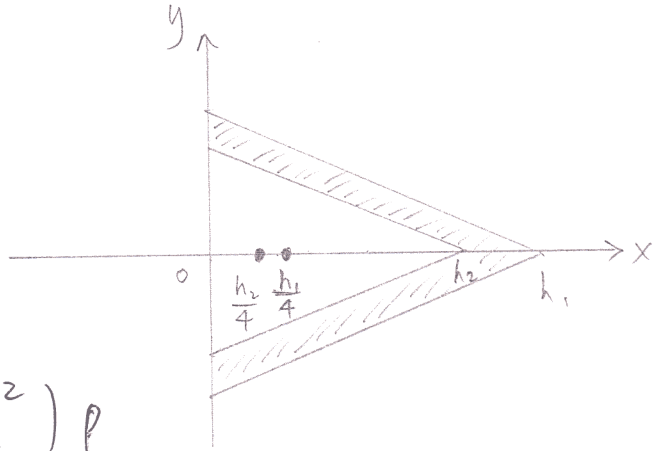
$$\bar{x} = \frac{3}{4}h \quad \text{from vertex}$$

i.e. The C.G. is $\frac{1}{4}h$ from the cone's base.

(b)

With thickness

$$C.G._H = \frac{M_1\left(\frac{h_1}{4}\right) - M_2\left(\frac{h_2}{4}\right)}{M_1 - M_2}$$



$$C.G._H = \frac{\frac{1}{4}\left(\frac{1}{3}\pi r_1^2 h_1^2 - \frac{1}{3}\pi r_2^2 h_2^2\right)\rho}{\left(\frac{1}{3}\pi r_1^2 h_1 - \frac{1}{3}\pi r_2^2 h_2\right)\rho}$$

$$= \frac{1}{4} \cdot \frac{r_1^2 h_1^2 - r_2^2 h_2^2}{r_1^2 h_1 - r_2^2 h_2}$$

$$= \frac{1}{4} \cdot \frac{h_1^2 \tan^2\theta\, h_1^2 - h_2^2 \tan^2\theta\, h_2^2}{h_1^2 \tan^2\theta\, h_1 - h_2^2 \tan^2\theta\, h_2}$$

$$C.G._H = \frac{1}{4} \cdot \frac{h_1^4 - h_2^4}{h_1^3 - h_2^3}$$

$$C.G._h = \underset{h_2 \to h_1}{Lt}\ C.G._H = \frac{1}{4} \underset{h_2 \to h_1}{Lt} \frac{h_1^4 - h_2^4}{h_1^3 - h_2^3}$$

$$C.G._h = \frac{1}{4} \underset{h_2 \to h_1}{Lt} \frac{(h_1 + h_2)(h_1^2 + h_2^2)}{h_1^2 + h_1 h_2 + h_2^2} = \frac{1}{4} \frac{(2h_1)(2h_1^2)}{3h_1^2} = \frac{h_1}{3}$$

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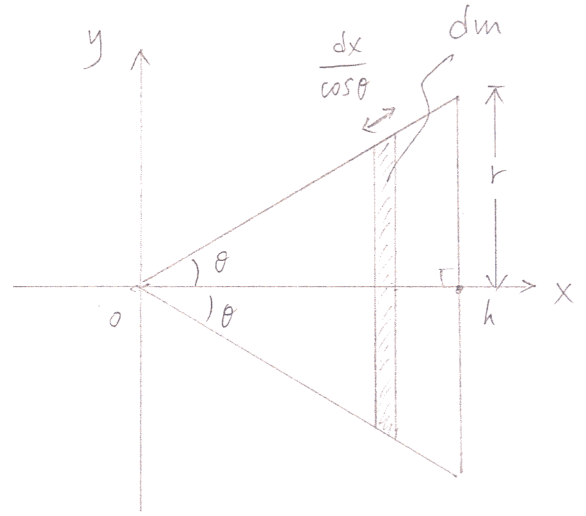
(c)

$$\bar{x}_h = \frac{\int x\, dm}{\int dm} \quad \text{where} \quad dm = 2\pi (x \tan\theta) \frac{dx}{\cos\theta} \sigma$$



$$\bar{x}_h = \frac{\int_0^h 2\pi \tan\theta \frac{\sigma}{\cos\theta} x^2\, dx}{\int_0^h 2\pi \tan\theta \frac{\sigma}{\cos\theta} x\, dx}$$

$$\bar{x}_h = \frac{\int_0^h x^2\, dx}{\int_0^h x\, dx}$$

$$\begin{cases} \sigma : \text{mass density} \\ \tan\theta = \frac{r}{h} = \text{constant} \end{cases}$$

$$\bar{x}_h = \frac{h^3/3}{h^2/2}$$

$\bar{x}_h = \frac{2}{3} h$ from vertex

$\therefore \bar{x}_h = \frac{2}{3} h$ from vertex $\Rightarrow$ C.G. of hollow cone is located $\frac{h}{3}$ from its base.

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