GENERAL RELATIVITY: THE GEODESICS OF THE SCHWARZSCHILD METRIC

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1. Introduction

Recall that the *exterior Schwarzschild metric* g defined on the 4-manifold $M = \mathbb{R} \times (\mathbb{R}^3 \backslash \overline{B}_{2m}) = \{(t, r, \theta, \phi) \colon\ r > 2m\}$ is given by:

$$g = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2).$$

It describes the gravitational field outside a spherically symmetric body of mass M. Four of the tests of general relativity are based on this metric and its geodesics: the gravitational redshift, the bending of light, the precession of the perihelion of Mercury, and the time delay of radar signals. In these lectures, we will study the null and timelike geodesics of the Schwarzschild metric, and describe the first three tests mentioned above.

The following simple result can be viewed as a special case of Nöther's Theorem.

Proposition 1. *Let* (M, g) *be a pseudo-Riemannian manifold, let* ξ *be a Killing vector field of* (M, g)*, and let* γ *be a geodesic of* (M, g)*. Then* $g(\xi, \dot\gamma)$ *is constant.*

Proof. Extend $\dot\gamma$ in a neighborhood of γ to a vector field X. Since $\mathcal{L}_\xi g = 0$, we have

$$2g(\nabla_\xi X, X) = \xi\big(g(X, X)\big) = 2g([\xi, X], X),$$

Thus:

$$X\big(g(\xi, X)\big) - g(\xi, \nabla_X X) = g(\nabla_X \xi, X) = g(\nabla_\xi X - [\xi, X], X) = 0.$$

Since along γ, we have $X = \dot\gamma$, and $\nabla_{\dot\gamma}\dot\gamma = 0$, we conclude $\dot\gamma\big(g(\xi, \dot\gamma)\big) = 0$. □

The Schwarzschild metric is rich in isometries; its isometry group is $G = \mathbb{R} \times SO(3)$. This, in conjunction with Proposition 1, will be used to integrate explicitly the geodesic equations.

The vector field $\partial_t = \partial/\partial t$ is a future directed timelike Killing field, which we call the *static Killing field*. Note that $g(\partial_t, \partial_t) = -(1 - 2m/r)$. The integral curves of ∂_t when reparameterized by proper time are the world lines of *static observers*, i.e. observers which are fixed as viewed from infinity.

2. The Gravitational Redshift

Light, as all electromagnetic waves, is modeled in relativity by solutions of the Maxwell equations. However, for high frequencies, the geometrical optics approximation is quite accurate. In this approximation, light signals are modeled by null geodesics. If a light signal is given by a null geodesic γ, then its frequency measured by an observer β receiving the signal at a point $p \in M$ is $\omega = -g(\dot\gamma, \dot\beta)|_p$. Consider now two static observers β_1 and β_2 in the Schwarzschild spacetime, and suppose that β_1 emits a light signal γ at point p_1 which is received by β_2 at point p_2. From the fact that both observers are static, it follows that

$$\dot\beta_j = \frac{\partial_t}{\sqrt{1-2m/r}}, \quad j = 1, 2.$$

By Proposition 1, we have $g(\partial_t, \dot\gamma)$ constant along γ, hence the *gravitational redshift factor*, defined to be one less than the ratio of the frequency of the light emitted ω_1 to the frequency of the light received ω_2, is:

$$\frac{\omega_1}{\omega_2} - 1 = \sqrt{\frac{1-2m/r_2}{1-2m/r_1}} - 1.$$

If the light is emitted closer to the center, where the gravitational field is stronger, and received further from the center, where the gravitational field is weaker, then the frequency is shifted toward the red. This effect has been measured to 1% accuracy in laboratory experiments on earth. The gravitational redshift of spectral lines from the Sun has also been observed, though with lesser accuracy due to other effects.

We make one last interesting observation related to the redshift. One can show rather easily that for any spherically symmetric perfect fluid body in equilibrium, there holds $M/R \leq 4/9$, regardless of the equation of state, where M is the total mass, R is the *area radius* $R = \sqrt{A/4\pi}$, and A is the surface area of the body. It follows that the maximum redshift factor, when $r_2 = \infty$ and $r_1 = R$, is 2. The redshift factor from quasars is commonly far larger than 2. This is presumably due to cosmological effects.

3. Integration of the Geodesic Equations

Let γ be any geodesic then $g(\dot\gamma, \dot\gamma)$ is constant along γ. Thus if γ is not null, we can assume that it is parameterized by proper time τ, i.e. that $g(\dot\gamma, \dot\gamma) = -1$. If γ is null, we assume that τ is any affine parameter along γ. Note that we still have the freedom to rescale τ this case. We also make the following simple observation.

Lemma 1. *Let (M, g) be a pseudo-Riemannian manifold, and let φ be an isometry of $(M.g)$. Let Σ be the set of fixed points of φ: $\{p \in M : \varphi(p) = p\}$. Then any geodesic γ initially on Σ and tangent to Σ remains in Σ. In particular, Σ is totally geodesic.*

Proof. Indeed, suppose to the contrary that $\gamma(\tau) \notin \Sigma$ for some τ. Then $\varphi(\gamma(\tau)) \neq \gamma(\tau)$. However, $\varphi \circ \gamma$ is a geodesic with the same initial conditions as γ, since $\varphi(\gamma(0)) = \gamma(0)$ and $\varphi(\dot\gamma(0)) = \dot\gamma(0)$. This violates the uniqueness theorem for geodesics. □

For the Schwarzschild solutions (M, g), the map $(t, r, \theta, \phi) \mapsto (t, r, \pi - \theta, \phi)$ is an isometry whose set of fixed points is the equatorial plane $\Sigma = \{(t, r, \theta, \phi)\colon\ \theta = \pi/2\}$. If γ is any geodesic then there is an isometry $\varphi \in SO(3)$ such that $\varphi(\gamma(0)) \in \Sigma$ and $\varphi(\dot\gamma(0)) \in T\Sigma$. By Lemma 1, it follows that $\varphi \circ \gamma$ remains in Σ. Therefore, it is sufficient to consider geodesics in the equatorial plane. The same result could have been obtained from Proposition 1.

Let $\gamma = (t, r, \pi/2, \phi)$ be a geodesic in the equatorial plane. Then (t, r, ϕ) is a critical point of the Lagrangian:

$$\mathcal{E} = \int \left\{ -\left(1 - \frac{2m}{r}\right)\dot t^2 + \left(1 - \frac{2m}{r}\right)^{-1}\dot r^2 + r^2\dot\phi^2 \right\} d\tau. \tag{1}$$

Using $\xi = \partial_t$ and $\xi = \partial_\phi$ in Proposition 1, we obtain two conserved quantities:

$$E = \left(1 - \frac{2m}{r}\right)\dot t \tag{2}$$

$$L = r^2\dot\phi, \tag{3}$$

which we call the *energy* and the *angular momentum* of γ respectively. If the angular momentum is zero, it follows that $\phi =$ constant. These geodesics, called *radial* geodesics, will be studied in Section 4. The last conserved quantity is obtained from the condition $g(\dot\gamma, \dot\gamma) =$ constant:

$$-\left(1 - \frac{2m}{r}\right)\dot t^2 + \left(1 - \frac{2m}{r}\right)^{-1}\dot r^2 + r^2\dot\phi^2 = -\epsilon, \tag{4}$$

where

$$\epsilon = \begin{cases} 0 & \text{if } \gamma \text{ is null;} \\ 1 & \text{if } \gamma \text{ is timelike.} \end{cases}$$

Substituting from (2) and (3) into (4), we obtain:

$$\dot r^2 + \left(1 - \frac{2m}{r}\right)\left(\frac{L^2}{r^2} + \epsilon\right) = E^2. \tag{5}$$

This is equivalent to the equation of motion for a particle of mass 1 and energy E^2 on a one-dimensional line in the effective potential $V = (1 - 2m/r)(L^2/r^2 + \epsilon)$. The qualitative analysis of these geodesics will be done in Section 5 for null geodesics $\epsilon = 0$, and in Section 6 for timelike geodesics $\epsilon = 1$.

4. Radial Geodesics

For radial geodesics, Equation (5) reduces to:

$$\dot{r}^2 + \epsilon\left(1 - \frac{2m}{r}\right) = E^2. \tag{6}$$

For null geodesics $\epsilon = 0$, we are free to rescale the affine parameter τ so that $E = 1$, and we obtain $\dot{r} = \pm 1$. Thus, after adjusting the origin of the affine parameter τ, we have $r = \pm\tau$. For *outgoing* null geodesics $\dot{r} = 1$, and Equation (2) now yields:

$$t - t_0 = \int \frac{d\tau}{1 - 2m/\tau} = \tau + 2m\log(\tau - 2m).$$

The outgoing null geodesics thus have the equation:

$$t - t_0 = r + 2m\log(r - 2m). \tag{7}$$

The *incoming* null geodesics can be obtained similarly, and have the equation:

$$t - t_0 = -\big(r + 2m\log(r - 2m)\big). \tag{8}$$

Note that all null geodesics reach $r = 2m$ within finite affine parameter even though Schwarzschild time t is infinite upon their arrival. Equations (7) and (8) are important when discussing extensions of the Schwarzschild spacetime beyond its *event horizon* $r = 2m$.

We now turn to the timelike radial geodesics. In this case, Equation (6) becomes:

$$\dot{r}^2 + \left(1 - \frac{2m}{r}\right) = E^2$$

It is interesting to note that the equation of motion in this case is $\ddot{r} = -m/r^2$ just as in the Newtonian case. The effective potential has no critical points, and therefore the motion is very simple. If $E^2 < 1$, then the orbit is a *crash orbit*, i.e. r climbs to a maximum at which point it turns around and decreases monotonically until $r = 2m$ within finite affine parameter. If $E^2 \geq 1$, then depending upon the initial direction, the orbit either escapes or crashes. We call these *crash/escape orbits*.

The equations of motion can be integrated in terms of quadratures:

$$\tau = \int \frac{dr}{\sqrt{E^2 - 1 + 2m/r}}$$

$$t = \int \frac{E^2 d\tau}{1 - 2m/r}.$$

Although, these integrals can be carried out explicitly, the resulting formulae bear no particular interest.

5. Null Geodesics and the Bending of Light

For null geodesics, we may as before, rescale the affine parameter to set $E = 1$. Equation (5) then becomes:

$$\dot{r}^2 + \frac{L^2}{r^2}\left(1 - \frac{2m}{r}\right) = 1.$$

The equations of motion again can be integrated in terms of quadratures:

$$\frac{d\tau}{dr} = \frac{1}{\sqrt{1 - (L^2/r^2)(1 - 2m/r)}} \tag{9}$$

$$\frac{dt}{d\tau} = \frac{1}{1 - 2m/r} \tag{10}$$

$$\frac{d\phi}{d\tau} = \frac{L}{r^2}. \tag{11}$$

The effective potential has one critical point at $r = 3m$. This correspond to an unstable circular orbit, a *trapped ray*. The potential has a maximum of $L^2/27m^2$ at this critical point. Hence an orbit initially in $r < 3m$ can escape if and only if $L^2 < 27m^2$, and conversely and orbit initially in $r > 3m$ will be not get trapped if and only if $L^2 > 27m^2$.

Consider now an orbit coming in from infinity, i.e. with $r \to \infty$ as $\tau \to -\infty$. Without loss of generality, assume that $L > 0$. Multiplying Equation (11) by Equation (9), we obtain

$$\frac{d\phi}{dr} = \frac{L}{r^2\sqrt{1 - (L^2/r^2)(1 - 2m/r)}}, \tag{12}$$

from which it follows immediately that $d\phi/dr \leq L/r^2$, and hence $\phi \to \phi_0 =$ constant as $\tau \to -\infty$. If $L^2 > 27m^2$, then the orbit will not get trapped, and $r \to \infty$ also as $\tau \to \infty$. Thus, $\phi \to \phi_1 =$ constant as $\tau \to +\infty$. We wish to compute the maximum deflection angle $\delta = \phi_1 - \phi_0 - \pi$ of the orbit from a linear orbit in flat space. It is more convenient to introduce the variable $u = 1/r$, and the parameter $a = 1/L$. This leads to the equation

$$\left(\frac{du}{d\phi}\right) + u^2(1 - 2mu) = a^2. \tag{13}$$

From the qualitative analysis, it is easily seen that the orbit will be symmetric about its *perihelion* where r will be at a minimum, say r_0, and consequently u at a maximum $u_0 = 1/r_0$. We point out that u_0 is the positive root of the cubic equation

$$a^2 - u^2 + 2mu^3 = 0. \tag{14}$$

In view of Equation (13), we now deduce that

$$\delta + \pi = 2\int_0^{u_0} \frac{du}{\sqrt{a^2 - u^2 + 2mu^3}}. \tag{15}$$

This is an elliptic integral, and cannot be carried out explicitly. However, we only need to compute the linear contribution of m to this integral. To this purpose, we make the substitutions $v = u/u_0$ and $\mu = u_0 m$ in (15). Then, noting that Equation (14) implies $a^2 = u_0^2(1 - 2\mu)$, we find

$$\delta = \int_0^1 \frac{dv}{\sqrt{1 - 2\mu - v^2 + 2\mu v^3}}.$$

Since clearly δ is only a function of μ, we have $d\delta = \delta_\mu d\mu = \delta_\mu(u_0\, dm + m\, du_0)$. When $m = 0$, we find

$$\delta_\mu(0) = 2\int_0^1 \frac{(1 - v^3)}{(1 - v^2)^{3/2}}\, dv = 4,$$

and also $d\mu = u_0\, dm$. Consequently, we obtain

$$d\delta|_{m=0} = 4u_0\, dm.$$

Thus, to first order in m, a null orbit passing at a perihelion of r_0 will experience a deflection in its azimuthal angle of

$$\delta \approx \frac{4Gm}{c^2 r_0},$$

where G is the gravitational constant and c the speed of light. In cgs units $G = 6.67 \times 10^{-8}\mathrm{cm}^3/\mathrm{g\,sec}^2$, and $c = 3 \times 10^{10}\mathrm{cm/sec}$. With $m = 2 \times 10^{33}\mathrm{g}$ and $r_0 = 7 \times 10^{10}\mathrm{cm}$ the mass and radius of the sun respectively, one obtains for an orbit which grazes the surface of the sun:

$$\delta \approx 0.85 \times 10^{-5},$$

or about 1.7 seconds of arc. This is within 1% of observed data.

6. Timelike Geodesics and Perihelion Precession

We now turn to the study of the timelike geodesics. Equation (4) now has the form

$$\dot{r}^2 + \left(1 - \frac{2m}{r}\right)\left(\frac{L^2}{r^2} + 1\right) = E^2.$$

Thus, as before, this is identical to the motion of a particle of mass 1 on a line in the effective potential $V(r) = 1 - 2m/r + L^2/r^2 - 2mL^2/r^3$. It is interesting to note that the only difference with the Newtonian case is the relativistic correction term $-2mL^2/r^3$. The qualitative analysis of this potential depends on the ratio L^2/m^2.

If $L^2 < 12m^2$, then V has no critical points and just as in the radial case, the analysis is simple: depending whether the energy $E^2 < 1$ or $E^2 \geq 1$ the orbit is either a crash or a crash/escape orbit.

If $L^2 = 12m^2$, the situation is the essentially the same, with the exception that there is now an unstable circular orbit at $r = 6m$, and an exceptional orbit with $E^2 = 8/9$ which spirals into this orbit.

If $L^2 > 12m^2$, there are two critical points $r_1 < 6m < r_2$, the roots of the equation $mr^2 - L^2 r + 3mL^2 = 0$. We will not proceed further with this analysis, but instead point out that the circular orbit at $r = r_2$ is now a stable circular orbit. Orbits with r initially close to r_2 and energy E^2 slightly larger than $V(r_2)$ will have r oscillating between $r_{\min}$ and $r_{\max}$. When L^2/m^2 is much larger than 12, than r_2 will be large, hence those orbits will have r large throughout and the relativistic correction term small. These orbit will be nearly elliptical. Consider such an orbit, with perihelion at $\tau = 0$ $\phi = 0$, $r = r_{\min}$. The orbit will reach perihelion again at some proper time $\tau = \tau_0$, and $\phi = \phi_0$ We now wish to compute the *perihelion precession* of such an orbit, i.e. $\psi = \phi_0 - 2\pi$.

We can write

$$\psi = 2\int_{r_{\min}}^{r_{\max}} \frac{L\,dr}{r^2\sqrt{E^2 - V(r)}} - 2\pi. \tag{16}$$

Introduce the variable $u = 1/r$, and the parameters $u_{\min} = 1/r_{\min}$ and $u_{\max} = 1/r_{\max}$. Then the integral in (16) can be rewritten as

$$\psi = 2\int_1^{\alpha} \frac{du}{\sqrt{(E^1-1)/L^2 + (2m/L^2)u - u^2 + 2mu^3}} - 2\pi. \tag{17}$$

Again, this is an elliptic integral, and cannot be evaluated explicitly. However, as before, it is sufficient to compute the linear contribution of m. For this purpose, introduce the parameters a and e by:

$$r_{\min} = a(1-e), \qquad r_{\max} = a(1+e),$$

and the variable $v = au$. In analogy with Newtonian mechanics, we call a and e the *eccentricity* and *semimajor axis* of the orbit respectively. With these substitution, the integral in (17) becomes

$$\psi = 2\int_{v_1}^{v_2} \frac{dv}{\sqrt{\mu(v_0 - v)(v - v_1)(v_2 - v)}} - 2\pi,$$

where $\mu = 2m/a$, $v_1 = 1/(1+e)$, $v_2 = 1/(1-e)$, and $v_0 + v_1 + v_2 = 1/\mu$. It is clear from this formula that ψ is a function of e and μ. It is easy to see that when $\psi = 0$ when $\mu = 0$, hence $\psi_e(0,e) = 0$. Differentiating with respect to μ at $\mu = 0$, we find:

$$\psi_\mu(0,e) = \int_{v_1}^{v_2} \frac{(v + v_1 + v_2)}{\sqrt{(v - v_1)(v_2 - v)}}\,dv = \frac{3\pi}{1-e^2}.$$

Therefore, we conclude that

$$d\psi|_{\mu=0} = \frac{3\pi}{1-e^2}\,d\mu.$$

Thus, for an orbit of ellipticity e and semimajor axis a, we have obtained to first order in m a perihelion precession rate

$$\psi \approx \frac{6\pi G m}{c^2 a(1-e^2)},$$

per revolution. For the orbit of Mercury around the sun, we have $e = 0.206$ and $a = 5.79 \times 10^{12}$cm, giving a perihelion advance $\psi \approx 5 \times 10^{-7}$ per revolution. The period T of Mercury can be obtained from Kepler's Third law

$$T^2 = \frac{4\pi^2 a^3}{Gm} = 5.7 \times 10^{13} \mathrm{sec}^2,$$

i.e. a period of about 7.6×10^6sec or 88 days, with about 415 revolutions per century. Thus, we get a perihelion precession rate of about 43 seconds of arc per century. This is exactly the observed value, after taking into account the Newtonian perturbations due to nearby planets. Einstein is quoted as having said that when he discovered this result he felt as if "the universe had whispered in his ear."