8

The Path Integral Formulation of Quantum Theory

We consider here an alternate formulation of quantum mechanics invented by Feynman in the forties.‡ In contrast to the Schrödinger formulation, which stems from Hamiltonian mechanics, the Feynman formulation is tied to the Lagrangian formulation of mechanics. Although we are committed to the former approach, we discuss in this chapter Feynman's alternative, not only because of its aesthetic value, but also because it can, in a class of problems, give the full propagator with tremendous ease and also give valuable insight into the relation between classical and quantum mechanics.

8.1. The Path Integral Recipe

We have already seen that the quantum problem is fully solved once the propagator is known. Thus far our practice has been to first find the eigenvalues and eigenfunctions of H, and then express the propagator $U(t)$ in terms of these. In the path integral approach one computes $U(t)$ directly. For a single particle in one dimension, the procedure is the following.

To find $U(x, t; x', t')$:

(1) Draw all paths in the x–t plane connecting (x', t') and (x, t) (see Fig. 8.1).
(2) Find the action $S[x(t)]$ for each path $x(t)$.
(3) $$U(x, t; x', t') = A \sum_{\text{all paths}} e^{iS[x(t)]/\hbar} \tag{8.1.1}$$

where A is an overall normalization factor.

‡ The nineteen forties that is, and in his twenties. An interesting account of how he was influenced by Dirac's work in the same direction may be found in his Nobel lectures. See, *Nobel Lectures—Physics*, Vol. III, Elsevier Publication, New York (1972).

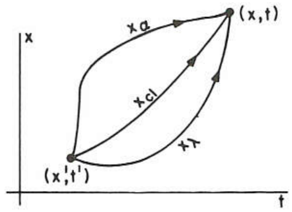

Figure 8.1. Some of the paths that contribute to the propagator. The contribution from the path $x(t)$ is $Z = \exp\{iS[x(t)]/\hbar\}$.

8.2. Analysis of the Recipe

Let us analyze the above recipe, postponing for a while the proof that it reproduces conventional quantum mechanics. The most surprising thing about it is the fact that every path, including the classical path, $x_{\text{cl}}(t)$, gets the same weight, that is to say, a number of unit modulus. How are we going to regain classical mechanics in the appropriate limit if the classical path does not seem favored in any way?

To understand this we must perform the sum in Eq. (8.1.1). Now, the correct way to sum over all the paths, that is to say, path integration, is quite complicated and we will discuss it later. For the present let us take the heuristic approach. Let us first pretend that the continuum of paths linking the end points is actually a discrete set. A few paths in the set are shown in Fig. 8.1.

We have to add the contributions $Z_\alpha = e^{iS[x_\alpha(t)]/\hbar}$ from each path $x_\alpha(t)$. This summation is done schematically in Fig. 8.2. Since each path has a different action, it contributes with a different phase, and the contributions from the paths essentially cancel each other, until we come near the classical path. Since S is stationary here, the Z's add constructively and produce a large sum. As we move away from $x_{\text{cl}}(t)$, destructive interference sets in once again. It is clear from the figure that $U(t)$ is dominated by the paths near $x_{\text{cl}}(t)$. Thus the classical path is important, not because it contributes a lot by itself, but because in its vicinity the paths contribute coherently.

How far must we deviate from x_{cl} before destructive interference sets in? One may say crudely that coherence is lost once the phase differs from the stationary value $S[x_{\text{cl}}(t)]/\hbar \equiv S_{\text{cl}}/\hbar$ by about π. This in turn means that the action for the coherence paths must be within $\hbar\pi$ of S_{cl}. For a macroscopic particle this means a very tight constraint on its path, since S_{cl} is typically $\simeq 1$ erg sec $\simeq 10^{27}\hbar$, while for an electron there is quite a bit of latitude. Consider the following example. A free particle leaves the origin at $t=0$ and arrives at $x=1$ cm at $t=1$ second. The classical path is

$$x = t \tag{8.2.1}$$

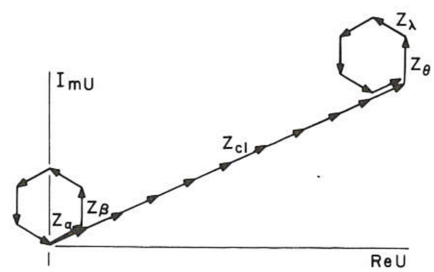

Figure 8.2. Schematic representation of the sum ΣZ_α. Paths near $x_{\text{cl}}(t)$ contribute coherently since S is stationary there, while others cancel each other out and may be ignored in the first approximation when we calculate $U(t)$.

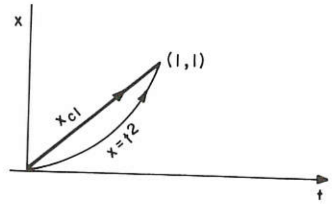

Figure 8.3. Two possible paths connecting (0, 0) and (1, 1). The action on the classical path $x=t$ is $m/2$, while on the other, it is $2m/3$.

Consider another path

$$x=t^2 \tag{8.2.2}$$

which also links the two space-time points (Fig. 8.3.)

For a classical particle, of mass, say 1 g, the action changes by roughly $1.6\times 10^{26}\hbar$, and the phase by roughly 1.6×10^{26} rad as we move from the classical path $x=t$ to the nonclassical path $x=t^2$. We may therefore completely ignore the nonclassical path. On the other hand, for an electron whose mass is $\simeq 10^{-27}$ g, $\delta S\simeq \hbar/6$ and the phase change is just around a sixth of a radian, which is well within the coherence range $\delta S/\hbar \lesssim \pi$. It is in such cases that assuming that the particle moves along a well-defined trajectory, $x_{\text{cl}}(t)$, leads to conflict with experiment.

8.3. An Approximation to $U(t)$ for a Free Particle

Our previous discussions have indicated that, to an excellent approximation, we may ignore all but the classical path and its neighbors in calculating $U(t)$. Assuming that each of these paths contributes the same amount $\exp(iS_{\text{cl}}/\hbar)$, since S is stationary, we get

$$U(t)=A'\, e^{iS_{\text{cl}}/\hbar} \tag{8.3.1}$$

where A' is some normalizing factor which "measures" the number of paths in the coherent range. Let us find $U(t)$ for a free particle in this approximation and compare the result with the exact result, Eq. (5.1.10).

The classical path for a free particle is just a straight line in the x–t plane:

$$x_{\text{cl}}(t'')=x'+\frac{x-x'}{t-t'}(t''-t') \tag{8.3.2}$$

corresponding to motion with uniform velocity $v=(x-x')/(t-t')$. Since $\mathscr{L}=mv^2/2$ is a constant,

$$S_{\text{cl}}=\int_{t'}^{t}\mathscr{L}\,dt''=\frac{1}{2}m\frac{(x-x')^2}{t-t'}$$