5) Let E be Cantor's familiar "middle thirds" set. Show that $m(E)=0$, even though E and $\mathbb{R}^1$ have the same cardinality.

Proof: Define the following linear transformations of $\mathbb{R}^1$ into $\mathbb{R}^1$ by

$$T_1(x)=3x \text{ and } T_2(x)=3\left(x-\tfrac{2}{3}\right).$$

Put $I_1=\left[0,\tfrac{1}{3}\right]$, $I_2=\left[\tfrac{2}{3},1\right]$, and $E_i=E\cap I_i$ for $i=1,2$. Then $T_i(E_i)=E$ for $i=1,2$ as, if I may take license, this is like unto prefacing the construction of the Cantor set with *Remove the open "middle third" of* $[0,1]$, *and then put it back; now construct the Cantor set as usual*. By Theorem 2.20 (e), $\exists\Delta(T_i)\in\mathbb{R}\ni m(T_i(A))=\Delta(T_i)m(A)$, $\forall$ Lebesgue measurable set $A\subset\mathbb{R}^1$, for $i=1,2$, and the constants $\Delta(T_i)$ are determined by, say $\Delta(T_i)=m(T_i(I_i))/m(I_i)=3$ for $i=1,2$, and thus $m(T_i(E_i))=3m(E_i)$ for $i=1,2$; but $T_i(E_i)=E$ for $i=1,2$, $E=E_1\cup E_2$, and $E_1\cap E_2=\varnothing$, so that

$$2m(E)=\sum_{i=1}^{2}m(T_i(E_i))=3\sum_{i=1}^{2}m(E_i)=3\left[m(E_1\cup E_2)+m(E_1\cap E_2)\right]=3\left[m(E)+m(\varnothing)\right]=3m(E).$$

Therefore $m(E)=0$.