We need to find self-similar solutions for the following PDE:

$$\frac{\partial h}{\partial t} = -h^2 \frac{\partial h}{\partial x} \tag{1}$$

with $-1 \leq x \leq 1, \quad t \geq 0.$

The initial condition is given by:

$$h(x,t) = 0.1 + 0.5(1 + cos(\pi x)) \tag{2}$$

The goal is to find special solutions of the normalized evolution equation, with the property that the solutions remain the same under a certain transformation of variables; these solutions are also referred to as "symmetry solutions". A very useful subclass of symmetry transformations involves a transformation of dependent and independent variables by scaling. We impose the following scaled variables:

$$\begin{cases} \hat{x} = \lambda x \\ \hat{t} = \lambda^{\alpha} t \\ \hat{h} = \lambda^{\beta} h \end{cases} \tag{3}$$

Thus, we transform the original PDE to a PDE with new the variables:

$$\frac{\partial h}{\partial t} = -h^2 \frac{\partial h}{\partial x} \quad \Longleftrightarrow \quad \frac{\partial \hat{h}}{\partial \hat{t}} = -\hat{h}^2 \frac{\partial \hat{h}}{\partial \hat{x}} \tag{4}$$

and we assume that the solution has the following form:

$$h(x,t) = t^{\beta/\alpha} H(\eta), \tag{5}$$

where η is the similarity variable.

Inserting the new variables in the PDE gives:

$$\begin{cases} \dfrac{\partial \hat{h}}{\partial \hat{t}} = \dfrac{\partial(\lambda^{\beta} h)}{\partial(\lambda^{\alpha} t)} = \lambda^{(\beta-\alpha)} \dfrac{\partial h}{\partial t} \\ \dfrac{\partial \hat{h}}{\partial \hat{x}} = \dfrac{\partial(\lambda^{\beta} h)}{\partial(\lambda x)} = \lambda^{(\beta-1)} \dfrac{\partial h}{\partial x} \end{cases} \tag{6}$$

In order for Eq. 4 to hold true, it is required that $\alpha = 1$. One can now determine the form of the self-similar ansatz by verifying that the ratio x/t does not change:

$$\begin{cases} \hat{x} = \lambda x \\ \hat{t} = \lambda t \end{cases} \quad \Longrightarrow \quad \lambda = \frac{\hat{x}}{x} = \frac{\hat{t}}{t} \quad \Longrightarrow \quad \frac{\hat{x}}{\hat{t}} = \frac{x}{t} = \eta \tag{7}$$

Note that the original PDE can be written in conservation law form with flux $h(x,t)$ as:

$$h_t + \left(\frac{h^3}{3}\right)_x = 0 \tag{8}$$

and we can therefore say that:

$$\frac{\partial}{\partial t}\left(\int_{-\infty}^{\infty} h(x,t) \quad dx\right) = 0 \tag{9}$$

Using Eq. 9, we obtain the following:

$$\frac{\partial}{\partial t}\left(t^{\beta}\int_{-\infty}^{\infty} H\left(\frac{x}{t}\right) \quad dx\right) = 0, \tag{10a}$$

$$\frac{\partial}{\partial t}\left(t^{(\beta+1)}\int_{-\infty}^{\infty} H(\eta) \quad d\eta\right) = 0, \tag{10b}$$

and therefore, $\beta = -1$.

Thus, know that the solution has takes the following form:

$$h(x,t) \quad = \quad t^{-1}H(\eta) \quad = \quad t^{-1}H\left(\frac{x}{t}\right) \tag{11}$$

We can now compute the terms in Eq. (1) as:

$$\frac{\partial h}{\partial x} = \quad t^{-1}\frac{\partial H}{\partial \eta}\frac{\partial \eta}{\partial x} = \quad t^{-2}H' \tag{12a}$$

$$\frac{\partial h}{\partial t} = \quad \left(-t^{-2}H\right) + \left(t^{-1}\frac{\partial H}{\partial \eta}\frac{\partial \eta}{\partial t}\right) = \quad \left(-t^{-2}H\right) + \left(-t^{-3}xH'\right) \tag{12b}$$

$$h^2 = t^{-2}H^2 \tag{12c}$$

Plugging Eq. (12a) - (12c) into Eq. (1) results in:

$$-H - \eta H' + \frac{H^2}{t^2}H' = 0 \tag{13}$$