PHYSICS 151 SOLUTIONS 5

Problem 1. Let $\vec{\mathbf{L}} = \sum_{j=1}^{N} \vec{L}^{(j)}$. Then

$$[\vec{\mathbf{L}}_a, \vec{\mathbf{L}}_b] = [\sum_{i=1}^{N} L_a^{(i)}, \sum_{j=1}^{N} L_b^{(j)}] = \sum_{i,j=1}^{N} [L_a^{(i)}, L_b^{(j)}] = \sum_{i} [L_a^{(i)}, L_b^{(i)}]\,,$$

where in the last equality, we used the fact that if $i \neq j$, the ith and the jth particles are independent, so for $i \neq j$, $[f(q_i, p_i), g(q_j, p_j)] = 0$ for any differentiable functions f, g. We now use the fact, as shown in class, that the angular momentum algebra is obeyed by each of the particles, so $[L_a^{(i)}, L_b^{(i)}] = \sum_{c=1}^{3} \epsilon_{abc} L_c^{(i)}$. Therefore,

$$[\vec{\mathbf{L}}_a, \vec{\mathbf{L}}_b] = \sum_{c=1}^{3} \epsilon_{abc} \vec{\mathbf{L}}_c\,.$$

Problem 2. We adopt the convention that a repeated index is assumed to be summed from 1 to 3, so $\vec{f} \cdot \vec{g} = f_j g_j$, for example. This convention is used throughout this problem. We first prove a few general facts that will be useful for several of the sub-parts, as well as in the rest of the course.

Fact 1. If $\vec{f}$ and $\vec{g}$ are two functions on phase space that transform as vectors under a rotation, then their dot product $\vec{f} \cdot \vec{g}$ transforms as a scalar. The cross product $\vec{f} \wedge \vec{g}$ transforms as a vector.

Proof. Then

$$[L_i, f_j g_j] = f_j [L_i, g_j] + [L_i, f_j] g_j = f_j \epsilon_{ijk} g_k + \epsilon_{ijk} f_k g_j = \epsilon_{ijk}(f_j g_k + f_k g_j) = 0.$$

The last expression vanishes because ϵ_{ijk} is completely antisymmetric, and it is contracted with something symmetric. There is an analogous argument for the cross product. □

Fact 2. Any analytic function of a scalar is also a scalar.

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Proof. Let $\phi(z)$ be an analytic function of one variable. Then $\phi(z)$ may be expressed as a convergent power series, $\phi(z) = a_0 + a_1 z + a_2 z^2 + \dots$. Let $f(x,p)$ be a function on phase space, such that $[L_i, f] = 0$ for all $i = 1,2,3$. Then, if n is a positive integer, it follows that $[L_i, f^n] = D_{L_i}(f^n) = n f^{n-1} D_{L_i} f = 0$. Therefore, $[L_i, a_n f^n] = 0$ for all n, hence $[L_i, \phi(f)] = 0$. □

Fact 3. Let $\vec{f}$ be nonzero and transform as a vector. The components f_i do not transform as scalars.

Proof. $[L_i, f_j] = \epsilon_{ijk} f_k$. If the right-hand side is zero for all $i = 1,2,3$ then $\vec{f} = 0$, which we ruled out by assumption. □

Fact 4. Given a nonzero $\vec{f}$ which transforms as a vector, one can define a *dual* object $f_{jk} := \epsilon_{jka} f_a$. This dual is antisymmetric ($f_{jk} = -f_{kj}$), and it transforms as a two-index tensor.

Conversely, a two-index antisymmetric tensor η_{jk} where $j,k = 1,2,3$ there are exactly 3 independent components. This tensor has a *dual* $\eta_a := \frac{1}{2}\epsilon_{ajk}\eta_{jk}$, which transforms as a vector.

Applying the duality twice returns the original vector or tensor.

Proof. We have

$$\epsilon_{ija}\epsilon_{ikb} f_{ab} = -\epsilon_{ija}\epsilon_{ikb}\epsilon_{bac} f_c = -\epsilon_{ija}(\delta_{ia}\delta_{kc} - \delta_{ic}\delta_{ka}) f_c = \epsilon_{ija}\delta_{ka} f_i = \epsilon_{ijk} f_i \,.$$

On the other hand, $[L_i, f_{jk}] = \epsilon_{ajk}[L_i, f_a] = \epsilon_{ajk}\epsilon_{iac} f_c = -\delta_{ji} f_k + \delta_{ki} f_j = \epsilon_{ijk} f_i$. Therefore, f_{ab} transforms as a two-index tensor.

□

Now, we can solve the problem more easily:

(i) Since $\vec{x}$ is a vector, Fact 1 $\Rightarrow \vec{x} \cdot \vec{x}$ is a **scalar**.

(ii) It is obvious that any linear combination of scalars is another scalar, since the Poisson bracket is linear; Fact 1 implies that each term here is a scalar. Therefore **the sum is a scalar**.

(iii) By fact 2, this is a **scalar**.

(iv) $\sum_j [L_i, x_j^3] = 3\epsilon_{ijk} x_k x_j^2 \neq 0$. Therefore $\sum_j x_j^3$ is **not a scalar (nor a vector nor a tensor)**.

(v) $\vec{x}$ is a **vector**.

(vi) $\vec{x} \wedge \vec{p}$ is a **vector** by Fact 1.

(vii) This is the 2-index tensor dual to the vector $\vec{x} \wedge \vec{p}$, as in Fact 4. Hence it's a **tensor**.

(viii) This is a two-index **tensor**. In fact, this tensor is called the *tensor product* of the vectors $\vec{x}$ and $\vec{p}$. One way to prove this is to check that the dual vector $v_a := \epsilon_{ajk} x_j p_k$ actually transforms as a vector, i.e. to check that $[L_i, v_a] = \epsilon_{iab} v_b$. One has $[L_i, v_a] = \epsilon_{ajk} \left(\epsilon_{ikl} x_j p_l + \epsilon_{ijl} x_l p_k \right) = x_i p_a - x_a p_i$ and

$$\epsilon_{iab} \epsilon_{bjk} x_j p_k = (\delta_{ij} \delta_{ak} - \delta_{ik} \delta_{aj}) x_j p_k = x_i p_a - x_a p_i \, .$$

(ix) This $\vec{f}$ has the form "scalar times vector," hence is a **vector**.

(x) By fact 2, this is a **scalar**.

(xi) To determine whether this is a two-index tensor, we can ignore the $\vec{x}^2$, which will transform as a scalar, and hence will drop out of all Poisson brackets. Now let's consider the right-hand side of the transformation equation for a 2-tensor.

$$\epsilon_{ija} \epsilon_{ikb} \delta_{ab} = \epsilon_{ija} \epsilon_{ika} = \delta_{ii} \delta_{jk} - \delta_{ik} \delta_{ji} = \delta_{jk} (1 - \delta_{ik}) \, .$$

If $j = k \neq i$, this expression is non-zero, while $[L_i, \delta_{jk}] = 0$. Therefore the expression is **not a 2-tensor**. However, each component **is a scalar**.

(xii) The Lenz vector points from the geometric center of the orbit to the attracting, central body, and has length $|\vec{A}| = mk\epsilon$. It is hence covariant under rotations, and therefore a **vector**.

(xiii) We saw in Problem 1 that this is a **vector**.