Solving Cubic Equations

by

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Solving Cubic Equations

Assume the original real number cubic equation is of the form:

$$ax^3 + bx^2 + cx + d = 0 \quad \text{where } a \neq 0.$$

We can factor out the leading coefficient a and re-write the polynomial equation in the form:

$$a[\,x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a}] = 0.$$

Since $a \neq 0$ we can divide by a and thus assume that our cubic is a monic polynomial of the form:

$$x^3 + Ax^2 + Bx + C = 0$$

where $A = \frac{b}{a}$ and $B = \frac{c}{a}$ and $C = \frac{d}{a}$.

Next we will make the change of variable $u = x + \frac{A}{3}$. We do this to derive a new cubic equation in which the squared term will be missing. It may not be obvious why or how this works so we must show the details. Note that $x = u - \frac{A}{3}$ so that when we substitute for x, the last cubic equation above becomes:

$$(u - \tfrac{A}{3})^3 + A(u - \tfrac{A}{3})^2 + B(u - \tfrac{A}{3}) + C = 0$$

$$u^3 - 3 \cdot u^2\left(\tfrac{A}{3}\right) + 3u\left(\tfrac{A}{3}\right)^2 - \left(\tfrac{A}{3}\right)^3 + Au^2 - A\tfrac{2A}{3}u + A\left(\tfrac{A}{3}\right)^2 + Bu - \tfrac{AB}{3} + C = 0$$

$$u^3 - Au^2 + u\tfrac{A^2}{3} - \tfrac{A^3}{27} + Au^2 - \tfrac{2A^2}{3}u + \tfrac{A^3}{9} + Bu - \tfrac{AB}{3} + C = 0$$

$$u^3 + u\tfrac{A^2}{3} - \tfrac{A^3}{27} - \tfrac{2A^2}{3}u + \tfrac{A^3}{9} + Bu - \tfrac{AB}{3} + C = 0$$

$$u^3 + \left[B - \tfrac{A^2}{3}\right]u + \left[\tfrac{2A^3}{27} - \tfrac{AB}{3} + C\right] = 0$$

The last equation means we can always reduce the original cubic equation to one that is of the simpler form:

$$u^3 - Mu - N = 0 \tag{$*$}$$

In turn, we temporarily switch back to using the x variable and re-write this last equation in the form $x^3 = Mx + N$. This represents the intersection in the xy-plane of the line $y = Mx + N$ with the graph of $y = x^3$. This last form is intellectually interesting where we consider the two cases of both positively and negatively sloped lines. See Figure 1 on the next page. A negatively sloped line will intersect $y = x^3$ in only one point while a positively sloped line *may* intersect $y = x^3$ in three points, but can be guaranteed to intersect $y = x^3$ in at least one point whose x-coordinate will be a real number.

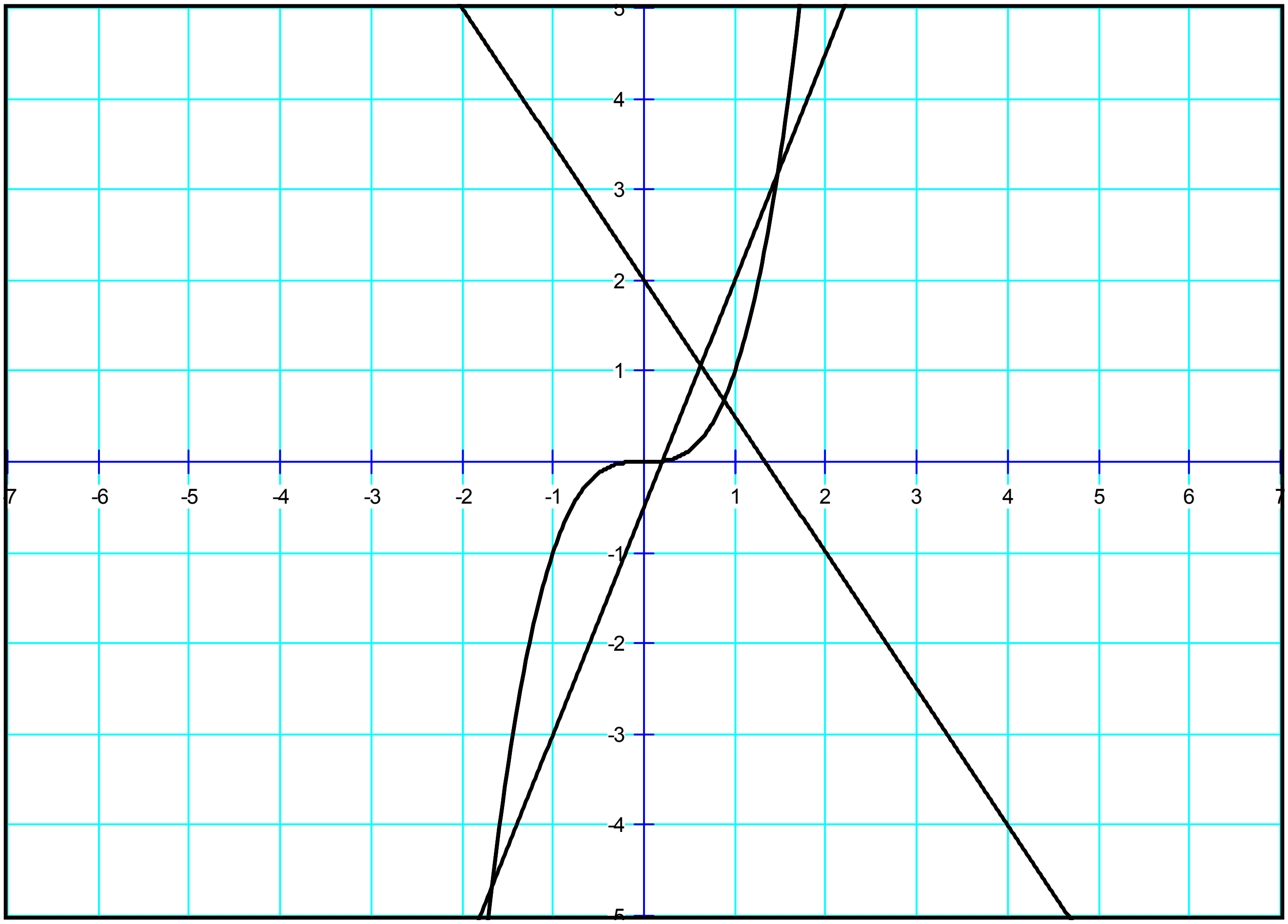

Figure 1. The graph of $y = x^3$ intersected by a line with negative slope and a line with positive slope.

Next, we state and prove a simple Lemma that will be of use in solving the above cubic equation $(*)$.

Lemma: If the sum S and product P of two numbers (either real or complex) are known, then the two numbers can be discovered as the two solutions to the quadratic equation in the variable w:

$$w^2 - Sw + P = 0$$

Proof: If we apply the quadratic formula we arrive at the two solutions.

$$w_1 = \frac{S + \sqrt{S^2 - 4P}}{2} \text{ and } w_2 = \frac{S - \sqrt{S^2 - 4P}}{2}$$

Now it is trivial that $w_1 + w_2 = \dfrac{S + \sqrt{S^2 - 4P}}{2} + \dfrac{S - \sqrt{S^2 - 4P}}{2} = \dfrac{2S}{2} = S$.

Moreover, we find the product can also be calculated as

$$w_1 \cdot w_2 = \left(\frac{S + \sqrt{S^2 - 4P}}{2}\right)\left(\frac{S - \sqrt{S^2 - 4P}}{2}\right) = \frac{S^2 - (S^2 - 4P)}{4} = \frac{4P}{4} = P.$$

The last technique we need is related to a simple identity where if p and q are any two numbers (again, either real or complex) then by the **Binomial Theorem** we can write:

$$(p+q)^3 = p^3 + 3p^2q + 3pq^2 + q^3 = p^3 + 3pq(p+q) + q^3$$

It takes a little imagination, but the above equation is significant because the middle part of the last expression involves both the product and sum of p and q and the whole equation involves the sum of the two numbers $p^3 + q^3$. We can re-write the above cubic identity one more time as:

$$(p+q)^3 - 3pq(p+q) - [p^3 + q^3] = 0 \qquad (**)$$

Now when we consider the form of the cubic equation:

$$u^3 - Mu - N = 0 \qquad (*)$$

we can begin to understand that it almost fits the form of the $(**)$ equation if we just let $u = p + q$. Then we can write:

$$u^3 - (3pq)u - [p^3 + q^3] = 0$$

$$u^3 - \left(3\sqrt[3]{p^3}\,\sqrt[3]{q^3}\right)y - [p^3 + q^3] = 0$$

If only we could find two numbers p^3 and q^3 such that $M = 3\sqrt[3]{p^3}\,\sqrt[3]{q^3}$ and $N = [p^3 + q^3]$ we could solve the equation. But if we know the number M then we know $\left(\frac{M}{3}\right)^3 = p^3q^3$ that should be the product of p^3 and q^3. If we know N then we know the sum $p^3 + q^3$. The subtle surprise here is that the two numbers we first need to find are actually p^3 and q^3 (not p and q) whose product p^3q^3 is known as $\left(\frac{M}{3}\right)^3$ and whose sum $p^3 + q^3$ is also known as N. To continue finding the solution we first apply the **Lemma** and setup and solve the quadratic:

$$w^2 - (p^3 + q^3)w + p^3q^3 = 0$$

$$w^2 - Nw + \left(\frac{M}{3}\right)^3 = 0$$

Then we would have: $\quad p^3 = \dfrac{N + \sqrt{N^2 - \frac{4M^3}{27}}}{2} \quad$ and $\quad q^3 = \dfrac{N - \sqrt{N^2 - \frac{4M^3}{27}}}{2}.$

In other words, one solution of the $(*)$ equation for u is given by:

$$u = p + q = \sqrt[3]{\frac{N + \sqrt{N^2 - \frac{4M^3}{27}}}{2}} + \sqrt[3]{\frac{N - \sqrt{N^2 - \frac{4M^3}{27}}}{2}}$$

We can see that when $N^2 - \frac{4M^3}{27} < 0$ then the above derivation leads to two complex numbers that are added together. However, those complex numbers are complex conjugates of one another and as we will see, their sum causes their imaginary parts to cancel and their real parts to double, resulting in a real number answer for u.

We will work a couple of numerical examples that will show how to handle the details.

Our first equation is: $x^3 - 15x - 4 = 0$. When we re-arrange this to get $x^3 = 15x + 4$ we can think where does the graph of the line $y = 15x + 4$ intersect the graph of $y = x^3$?

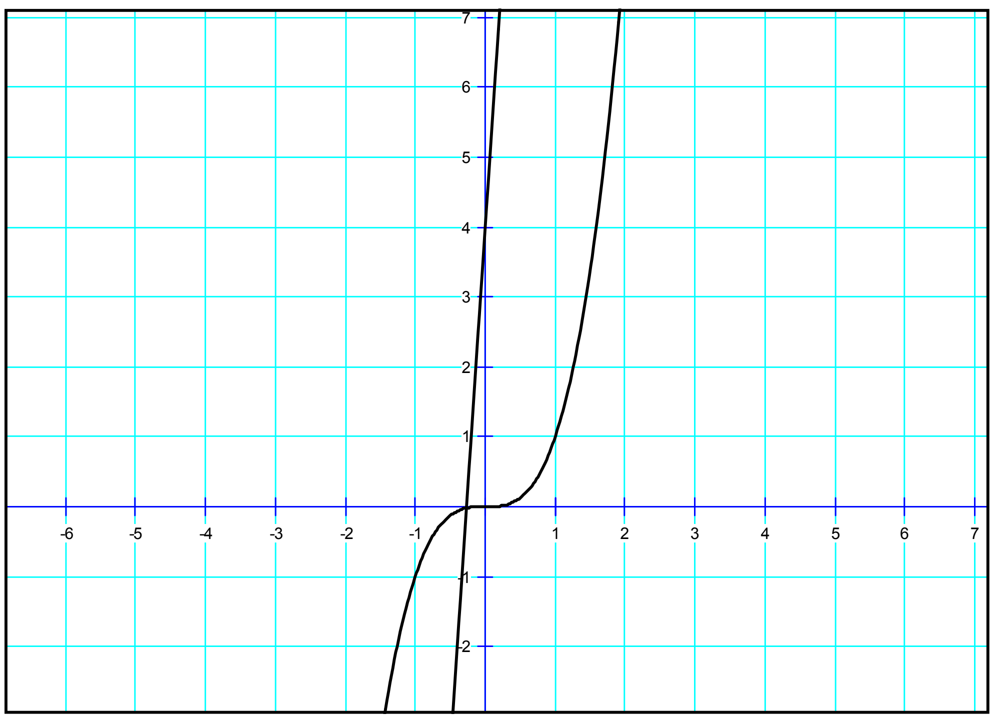

Figure 2. An intersection point of the line $y = 15x + 4$ with the cubic $y = x^3$.

In Figure 2 we can see a point of intersection where x should be a small negative number between -1 and 0, but closer to 0. We might estimate x is near $-\frac{1}{3}$.

Now consider the original cubic equation in the special form:

$$x^3 - 15x - 4 = 0$$

Now we can see that $M = 15$ and $N = 4$. Then, $\left(\frac{M}{3}\right)^3 = 5^3 = 125$. $\frac{4M^3}{27} = 500$.

$$\sqrt{N^2 - \frac{4M^3}{27}} = \sqrt{16 - 500} = 2\sqrt{4 - 125} = 2\sqrt{-121} = 22i$$

$$p^3 = \frac{N + \sqrt{N^2 - \frac{4M^3}{27}}}{2} = \frac{4 + 22i}{2} = 2 + 11i$$

$$q^3 = \frac{N - \sqrt{N^2 - \frac{4M^3}{27}}}{2} = \frac{4 - 22i}{2} = 2 - 11i$$

Finally, $x = p + q = \sqrt[3]{2 + 11i} + \sqrt[3]{2 - 11i}$

Now it is easiest to find the cube roots of complex numbers by writing those numbers in polar form. $a + bi = re^{i\theta}$ where $r = \sqrt{a^2 + b^2}$ and $\theta = tan^{-1}\left(\frac{b}{a}\right)$. Note that $e^{i\theta}$ is always a point on the unit circle by Euler's Identity. $e^{i\theta} = cos(\theta) + i \cdot sin(\theta)$. In polar form, the cube root of a complex number has an angle that is $\frac{1}{3}$ of the angle for the original complex number and it has a radius that is the cube root of the original radius. These facts are easily derived from DeMoivre's Theorem. In order to derive exact values of $cosine$ and $sine$ we would like to have an exact $\frac{1}{3}$-angle formula for the $cosine$ function, but this is difficult to come by, if not mathematically impossble to do exactly as we would like!

The complex number $2 + 11i = \sqrt{125}e^{i \cdot tan^{-1}\left(\frac{11}{2}\right)}$ so its cube root is $\sqrt[3]{\sqrt{125}}\, e^{i \cdot \frac{1}{3} tan^{-1}\left(\frac{11}{2}\right)} =$

$\sqrt[3]{\sqrt{125}}\, \left\{cos\left(\frac{1}{3}tan^{-1}\left(\frac{11}{2}\right)\right) + i \cdot sin\left(\frac{1}{3}tan^{-1}\left(\frac{11}{2}\right)\right)\right\} = p.$

Similarly, $\sqrt[3]{2 - 11i} = \sqrt[3]{\sqrt{125}}\, \left\{cos\left(\frac{1}{3}tan^{-1}\left(\frac{-11}{2}\right)\right) + i \cdot sin\left(\frac{1}{3}tan^{-1}\left(\frac{-11}{2}\right)\right)\right\} =$

$\sqrt[3]{\sqrt{125}}\, \left\{cos\left(\frac{1}{3}tan^{-1}\left(\frac{11}{2}\right)\right) - i \cdot sin\left(\frac{1}{3}tan^{-1}\left(\frac{11}{2}\right)\right)\right\} = q.$

Thus when we add these two cube roots, $p + q$, the imaginary parts cancel and the real parts double. At this point we discover that $\theta = tan^{-1}\left(\frac{11}{2}\right)$ or $tan(\theta) = \frac{11}{2}$. We can easily find that $cos(\theta) = \frac{2}{5\sqrt{5}}$ and $sin(\theta) = \frac{11}{5\sqrt{5}}$. However, it is not easy to find the exact cosine value of $\frac{1}{3}$ of the angle θ. With a calculator we discover that $cos^2\left(\frac{\theta}{3}\right) = 0.8$ so $cos\left(\frac{\theta}{3}\right) = \sqrt{0.8}$.

Now we can calculate x exactly! $x = 2 \cdot \sqrt[3]{\sqrt{125}}\, cos\left(\frac{1}{3}tan^{-1}\left(\frac{11}{2}\right)\right) = 2 \cdot 125^{\frac{1}{6}} \cdot cos\left(\frac{1}{3}tan^{-1}\left(\frac{11}{2}\right)\right) =$

$2\sqrt{5} \cdot \sqrt{0.8} = 2\sqrt{5(0.8)} = 2\sqrt{4} = 2 \cdot 2 = 4.$

Clearly we can see that $x = 4$ satisfies the equation: $x^3 = 15x + 4$. Thus we have found one real solution to the cubic equation $x^3 - 15x - 4 = 0$. By Synthetic Division we can deflate this polynomial to derive the quadratic factor.

$$\begin{array}{cccc|l} & & & & \underline{4} \\ 1 & 0 & -15 & -4 & \\ & 4 & 16 & 4 & \\ \hline 1 & 4 & 1 & 0 & \end{array}$$

Now we can find the other two real roots by solving the quadratic: $x^2 + 4x + 1 = 0$. The other two solutions are thus

$$x = \frac{-4 \pm \sqrt{16 - 4}}{2} = \frac{-4 \pm \sqrt{12}}{2} = \frac{-4 \pm 2\sqrt{3}}{2} = -2 \pm \sqrt{3}$$

We should note that finding these other two real solutions would not have been easy had we not found the one real solution first that required handling complex numbers.

Moreover, since $\sqrt{3} \approx 1.732$ we can reason that the root we saw in Figure 2 is really the number $-2 + \sqrt{3} \approx -0.268$. The other irrational root $-2 - \sqrt{3}$ is nowhere apparent in Figure 2, nor is the root $x = 4$ apparent in Figure 2.

Here's another numerical example that will lead us through slightly more messy details involving fractions and radicals with fractions.

Solve the equation: $2x^3 + 10x^2 + 8x - 4 = 0$.

This first becomes $x^3 + 5x^2 + 4x - 2 = 0$ after we divide by 2. Now we make a change of variable and let $x = u - \frac{5}{3}$.

$$\left(u - \tfrac{5}{3}\right)^3 + 5\left(u - \tfrac{5}{3}\right)^2 + 4\left(u - \tfrac{5}{3}\right) - 2 = 0.$$

$$u^3 - 3u^2\left(\tfrac{5}{3}\right) + 3u\left(\tfrac{5}{3}\right)^2 - \left(\tfrac{5}{3}\right)^3 + 5u^2 - \tfrac{50}{3}u + \tfrac{125}{9} + 4u - \tfrac{20}{3} - 2 = 0$$

$$u^3 - 5u^2 + \tfrac{25}{3}u - \tfrac{125}{27} + 5u^2 - \tfrac{50}{3}u + \tfrac{125}{9} + 4u - \tfrac{20}{3} - 2 = 0$$

$$u^3 - \tfrac{25}{3}u + 4u = \tfrac{125}{27} - \tfrac{125}{9} + \tfrac{20}{3} + 2$$

$$u^3 = \tfrac{25}{3}u - \tfrac{12}{3}u + \tfrac{125}{27} - \tfrac{375}{27} + \tfrac{180}{27} + \tfrac{54}{27}$$

$$u^3 = \tfrac{13}{3}u + \tfrac{-16}{27}$$

$$u^3 - \frac{13}{3}u + \frac{16}{27} = 0$$

Now we easily see that $M = \frac{13}{3}$ and $N = \frac{-16}{27}$. Then,

$$\sqrt{N^2 - \frac{4M^3}{27}} = \sqrt{\frac{256}{729} - \frac{4 \cdot 2197}{729}} = \sqrt{\frac{-8532}{729}} = \frac{6\sqrt{237}}{27}i.$$

$$\frac{N \pm \sqrt{N^2 - \frac{4M^3}{27}}}{2} = \frac{\frac{-16}{27} \pm \frac{6\sqrt{237}}{27}i}{2} = \frac{-8}{27} \pm \frac{3\sqrt{237}}{27}i$$

$$u = p + q = \sqrt[3]{\frac{-8}{27} + \frac{3\sqrt{237}}{27}i} + \sqrt[3]{\frac{-8}{27} - \frac{3\sqrt{237}}{27}i}$$

Again we have a sum of cube roots of complex number conjugates so the real parts double and the imaginary parts cancel. The radius of the complex number $\frac{-8}{27} + \frac{3\sqrt{237}}{27}i$ is :

$$\sqrt{\frac{64}{729} + \frac{9 \cdot 237}{729}} = \sqrt{\frac{64}{729} + \frac{2133}{729}} = \sqrt{\frac{2197}{729}} = \frac{13\sqrt{13}}{27}.$$

The polar angle of $\frac{-8}{27} + \frac{3\sqrt{237}}{27}i$ is : $\theta = tan^{-1}\left(\frac{\frac{3\sqrt{237}}{27}}{\frac{-8}{27}}\right) = tan^{-1}\left(\frac{-3\sqrt{237}}{8}\right).$

We can see this complex number lies in the second quadrant, a little left of the y-axis.

In degrees the θ-angle value for the complex number $\frac{-8}{27} + \frac{3\sqrt{237}}{27}i$ is:

$$\theta \approx -80.1728173731^{\circ} + 180^{\circ} \approx 99.8271826269^{\circ}.$$

Then $cos\left(\frac{\theta}{3}\right) \approx 0.836039871261$. Now we can try to calculate or at least approximate u.

$u = 2 \cdot \sqrt[3]{\frac{13\sqrt{13}}{27}} \cdot cos\left(\frac{1}{3}tan^{-1}\left(\frac{3\sqrt{237}}{-8}\right)\right) \approx 2 \cdot 1.20185042515 \cdot 0.836039871261 \approx 2.00958974944.$

Finally, the x-value we want is given by: $x = u - \frac{5}{3} \approx 0.34292308277.$

Alternative Methods Using Trigonometric and Hyperbolic Functions

One alternative method for solving any cubic equation in the form $x^3 + ax + b = 0$ is to exploit the fundamental trigonometric identity that $4 \cdot cos^3(\theta) - 3 \cdot cos(\theta) = cos(3 \cdot \theta)$. To do this we will temporarily assume $a < 0$ and we will later make another assumption regarding the size of b in relation to the size of a.

To get started we replace x in the original equation with the expression $k \cdot cos(\theta)$ where the constant k is yet to be determined.

$$k^3 cos^3(\theta) + akcos(\theta) + b = 0$$

Now multiply through by $\frac{4}{k^3}$ to get the equation in the form:

$$4cos^3(\theta) + \frac{4a}{k^2} cos(\theta) = \frac{-4b}{k^3}$$

Now we choose k so that the coefficient on the seconds term equals -3. $\frac{4a}{k^2} = -3$.

Choose $k = \sqrt{\frac{4a}{-3}}$. When $a < 0$ then k is well-defined. Then we have :

$$4cos^3(\theta) - 3cos(\theta) = \frac{-4b}{k^3}$$

Now we further assume $-1 \leq \frac{-4b}{k^3} \leq +1$. Then we can continue by setting $cos(3\theta) = \frac{-4b}{k^3}$ and finding all the possible answers for the angle θ. The three answers to the original cubic equation are simply $x = kcos(\theta)$ where we note that even though there may be infinitely many answers for θ all but three of those repeat the same $cos(\theta)$ values.

We now give an example using this new method.

Solve the equation $x^3 = 6x + 4$.

$$x^3 - 6x = 4$$

$$k^3 cos^3(\theta) - 6kcos(\theta) = 4$$

$$4cos^3(\theta) - \frac{4 \cdot 6}{k^2} cos(\theta) = \frac{4 \cdot 4}{k^3}$$

Now we choose k such that $\frac{24}{k^2} = 3$. In other words, choose $k = \sqrt{8}$.

$$4cos^3(\theta) - 3cos(\theta) = \frac{16}{8\sqrt{8}}$$

$$4cos^3(\theta) - 3cos(\theta) = \frac{16\sqrt{8}}{64} = \frac{\sqrt{8}}{4} = \frac{2\sqrt{2}}{4} = \frac{\sqrt{2}}{2}.$$

$$4cos^3(\theta) - 3cos(\theta) = \frac{\sqrt{2}}{2}.$$

Now we set $cos(3\theta) = \dfrac{\sqrt{2}}{2}$ and try to determine the exact value of $cos(\theta)$.

Then we know $3\theta = \dfrac{\pi}{4} \pm 2\pi n$ or $3\theta = \dfrac{7\pi}{4} \pm 2\pi n$.

$$\theta = \frac{\pi}{12} \pm \frac{2\pi}{3}n \text{ or } \theta = \frac{7\pi}{12} \pm \frac{2\pi}{3}n.$$

In terms of degrees we find there are three distinct cosine values.

We can write $x = \sqrt{8}cos(15^\circ)$ or $x = \sqrt{8}cos(135^\circ)$ or $x = \sqrt{8}cos(225^\circ)$.

It can be shown that $cos(15^\circ) = cos(45^\circ - 30^\circ) = cos(45^\circ)cos(30^\circ) + sin(45^\circ)sin(30^\circ) = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = \frac{\sqrt{6}+\sqrt{2}}{4} \approx 0.965925826$.

Thus for our first x we have $x = \sqrt{8} \cdot \frac{\sqrt{6}+\sqrt{2}}{4} = \frac{\sqrt{48}+\sqrt{16}}{4} = \frac{4\sqrt{3}+4}{4} = \sqrt{3} + 1 \approx 2.73205080757$.

For our second x it can be shown $cos(105^\circ) = cos(135^\circ - 30^\circ) = cos(135^\circ)cos(30^\circ) + sin(135^\circ)sin(30^\circ) = \frac{-\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = \frac{-\sqrt{6}+\sqrt{2}}{4} \approx -0.258819045102$.

Thus for our second x we have $x = \sqrt{8} \cdot \frac{-\sqrt{6}+\sqrt{2}}{4} = \frac{-\sqrt{48}+\sqrt{16}}{4} = \frac{-4\sqrt{3}+4}{4} = -\sqrt{3} + 1 \approx -0.73205080757$.

Our third x value is the easiest. $cos(225^\circ) = -\frac{\sqrt{2}}{2}$ so $x = \sqrt{8} \cdot \left(\frac{-\sqrt{2}}{2}\right) = \frac{-\sqrt{16}}{2} = \frac{-4}{2} = -2$.

Now we come back and explain how to handle the case where $a > 0$ in the cubic equation $x^3 + ax + b = 0$. In this case we can exploit the hyperbolic trigonometric identity that says $4 \cdot sinh^3(\theta) + 3 \cdot sinh(\theta) = sinh(3\theta)$. If you understood the case for using $cos(\theta)$ above then using the hyperbolic sine function will be just as simple for you to apply. Just mimic the same steps but substitute $x = k \cdot sinh(\theta)$ and later decide how to set the value for k.

The other cases we couldn't handle using $cos(\theta)$ is when $a < 0$ but the fraction $\frac{-4b}{k^3}$ is such that $\frac{-4b}{k^3} < -1$ or $\frac{-4b}{k^3} > +1$. In these cases we can use the hyperbolic cosine function and exploit the identity that says $4 \cdot cosh^3(\theta) - 3 \cdot cosh(\theta) = cosh(3\theta)$. Again, if you followed how to use the regular cosine function then using $cosh(\theta)$ should be easy.

Historically the Italian Renaissance mathematician del Ferro published his results in 1515. Thus we can understand most of this material is several hundred years old. Other Italian Renaissance mathematicians who contributed to the results discussed in this paper were Tartaglia and Cardano (1545) and Bombelli (1572). Francois Viete published his results in 1591. Solving cubic equations was responsible for the development of algebra in Europe which was further stimulated by the work to find solutions to polynomial equations of higher degrees. Later both Euler and Gauss contributed to popularizing the use of complex numbers.

After reading this paper you may wish to read another paper by the same author that is titled **The Pure Cubic Polynomial and Intersections With Linear Functions.**

Just visit **http://homepage.smc.edu/kennedy_john**