Solving Quartic Equations

by

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Solving Quartic Equations

Assume the original 4th degree or quartic equation is of the form:

$$ax^4 + bx^3 + cx^2 + dx + e = 0 \quad \text{where } a \neq 0.$$

We can factor out the leading coefficient a and rewrite the polynomial equation in the form:

$$a[\, x^4 + \frac{b}{a}x^3 + \frac{c}{a}x^2 + \frac{d}{a}x + \frac{e}{a}] = 0.$$

Since $a \neq 0$ we can divide by a and thus assume that our quartic is a monic polynomial of the form:

$$x^4 + Bx^3 + Cx^2 + Dx + E = 0$$

where $B = \frac{b}{a}$ and $C = \frac{c}{a}$ and $D = \frac{d}{a}$ and $E = \frac{e}{a}$.

Next we will make the change of variable $y = x + \frac{B}{4}$. We do this to derive a new quartic equation in which the x^3 term will be missing. It not be obvious why or how this works so we must show the details.

Note that $x = y - \frac{B}{4}$ so that the above equation becomes:

$$(y - \tfrac{B}{4})^4 + B(y - \tfrac{B}{4})^3 + C(y - \tfrac{B}{4})^2 + D(y - \tfrac{B}{4}) + E = 0$$

$$\begin{gathered} y^4 + 4 \cdot y^3\big(\tfrac{-B}{4}\big) \; + 6y^2\big(\tfrac{-B}{4}\big)^2 + 4y\big(\tfrac{-B}{4}\big)^3 + \big(\tfrac{-B}{4}\big)^4 + \\ By^3 + 3By^2\big(\tfrac{-B}{4}\big) + 3By\big(\tfrac{-B}{4}\big)^2 + B\big(\tfrac{-B}{4}\big)^3 + \\ + Cy^2 - \tfrac{BC}{2}y + \tfrac{B^2C}{16} + Dy - \tfrac{BD}{4} + E = 0 \end{gathered}$$

$$\begin{gathered} y^4 - By^3 \; + \Big(\tfrac{3B^2}{8}\Big)y^2 + \Big(\tfrac{-B^3}{16}\Big)y + \Big(\tfrac{B^4}{256}\Big) + \\ By^3 + \Big(\tfrac{-3B^2}{4}\Big)y^2 + \Big(\tfrac{3B^3}{16}\Big)y + \Big(\tfrac{-B^4}{64}\Big) + \\ + Cy^2 - \tfrac{BC}{2}y + \tfrac{B^2C}{16} + Dy - \tfrac{BD}{4} + E = 0 \end{gathered}$$

$$y^4 + \Big(\tfrac{-3B^2}{8} + C\Big)y^2 + \Big(\tfrac{B^3}{8} - \tfrac{BC}{2} + D\Big)y + \Big(\tfrac{-3B^4}{256} + \tfrac{B^2C}{16} - \tfrac{BD}{4} + E\Big) = 0$$

This shows that we can reduce our quartic equation to one that is of the form:

$$y^4 = My^2 + Ny + P$$

Theoretically, to solve this equation we must find all points of intersection of the quartic polynomial y^4 with a general quadratic. Another paper by this author discusses the theoretical basis for finding these solutions.

Now we continue by positioning P on the other side of the equation:

$$y^4 - P = My^2 + Ny$$

We will continue by essentially completing the square on the left side. To get the form we need, we will add $2\sqrt{-P}\,y^2$ in the middle of both sides of the last equation. To do this we temporarily assume $P < 0$. Then we can continue and write:

$$y^4 + 2\sqrt{-P}\,y^2 - P = My^2 + 2\sqrt{-P}\,y^2 + Ny$$

$$\left(y^2 + \sqrt{-P}\right)^2 = \left(M + 2\sqrt{-P}\right)y^2 + Ny$$

Next, we introduce an auxiliary variable z that will be determined later by adding the term $2\left(y^2 + \sqrt{-P}\right)z + z^2$ to both sides of the last equation.

$$\left(y^2 + \sqrt{-P}\right)^2 + 2\left(y^2 + \sqrt{-P}\right)z + z^2 = \left(M + 2\sqrt{-P}\right)y^2 + Ny + 2\left(y^2 + \sqrt{-P}\right)z + z^2$$

Then we can again complete the square on the left side and write:

$$\left[\left(y^2 + \sqrt{-P}\right) + z\right]^2 = \left(M + 2\sqrt{-P}\right)y^2 + Ny + 2\left(y^2 + \sqrt{-P}\right)z + z^2$$

Now we will choose any value of z that will also make the right side into a perfect square. However, we note that the right side is also a quadratic expression in the variable y.

$$\left[\left(y^2 + \sqrt{-P}\right) + z\right]^2 = \left(2\sqrt{-P} + M + 2z\right)y^2 + Ny + \left(2\sqrt{-P}\,z + z^2\right)$$

Now specially writing the right side as a perfect square is the same as thinking that the quadratic on the right has a repeated root and we know that can happen if and only if the discriminant of the quadratic is zero. Thus we choose z so that the discriminant $(N)^2 - 4\left(2\sqrt{-P} + M + 2z\right)\left(2\sqrt{-P}\,z + z^2\right) = 0$. This of course means z is a solution to some cubic polynomial in z because to solve that equation is to solve the cubic: $[-8]z^3 + \left[-4M - 24\sqrt{-P}\right]z^2 + \left[16P - 8M\sqrt{-P}\,\right]z + [N^2\,] = 0$.

Then we can continue and show the right side with a perfect square that will be in the form:

$$\left[\left(y^2 + \sqrt{-P}\right) + z\right]^2 = (Sy + T)^2$$

Now we can finish solving the quartic by taking square roots on both sides of the last equation.

$$\left[\left(y^2 + \sqrt{-P}\right) + z\right] = \pm\,(Sy + T)$$

We can try to temporarily ignore one of the signs on the right and rewrite this equation as:

$$y^2 + Sy + \left(\sqrt{-P} + z + T\right) = 0$$

Since this last equation is a quadratic in the variable y we will find two solutions to the original quartic equation if we can always guarantee finding two solutions to this last quadratic.

The astute reader will realize this depends in part on being able to always find an appropriate z-value. That of course requires solving the above cubic polynomial in z. Such a polynomial can always be guaranteed to have at least one real solution.

Now let's see an example of how we might apply the above techniques. The equation to be solved is:

$$x^4 + 4x^3 + 2x^2 - 3x - 1 = 0$$

We substitute $x = y - 1$.

$$(y-1)^4 + 4(y-1)^3 + 2(y-1)^2 - 3(y-1) - 1 = 0$$

$$y^4 - 4y^3 + 6y^2 - 4y + 1 + 4y^3 - 12y^2 + 12y - 4 + 2y^2 - 4y + 2 - 3y + 3 - 1 = 0$$

$$y^4 - 4y^2 + y + 1 = 0$$

$$y^4 + 1 = 4y^2 - y$$

$$y^4 + 2y^2 + 1 = 6y^2 - y$$

$$(y^2+1)^2 = 6y^2 - y$$

$$(y^2+1)^2 + 2(y^2+1)z + z^2 = 6y^2 - y + 2(y^2+1)z + z^2$$

$$\left[(y^2+1) + z\right]^2 = (6+2z)y^2 - y + (z^2 + 2z)$$

Now we compute the discriminant for the quadratic in y on the right and we set that discriminant equal to zero.

$$1 - 4(6+2z)(z^2+2z) = 0$$

$$-8z^3 - 40z^2 - 48z + 1 = 0$$

Now we must solve this cubic for a real value of z. The technique for solving a cubic is in another paper, but we begin by dividing by -8 to first make a leading coefficient of 1 on the z^3 term.

$$z^3 + 5z^2 + 6z - \tfrac{1}{8} = 0$$

Now make the change of variable $w = z + \frac{5}{3}$, or let $z = w - \frac{5}{3}$.

$$\left(w - \tfrac{5}{3}\right)^3 + 5\left(w - \tfrac{5}{3}\right)^2 + 6\left(w - \tfrac{5}{3}\right) - \tfrac{1}{8} = 0$$

$$w^3 - 3\left(\tfrac{5}{3}\right)w^2 + 3\left(\tfrac{5}{3}\right)^2 w - \left(\tfrac{5}{3}\right)^3 + 5w^2 - \tfrac{50}{3}w + \tfrac{125}{9} + 6w - 10 - \tfrac{1}{8} = 0$$

$$w^3 - 5w^2 + \tfrac{25}{3}w - \tfrac{125}{27} + 5w^2 - \tfrac{50}{3}w + \tfrac{125}{9} + 6w - \tfrac{81}{8} = 0$$

$$w^3 - \tfrac{125}{27} - \tfrac{25}{3}w + \tfrac{125}{9} + 6w - \tfrac{81}{8} = 0$$

$$w^3 - \tfrac{7}{3}w + \tfrac{-187}{216} = 0$$

$$w^3 = 3 \cdot \tfrac{7}{9}w + 2 \cdot \tfrac{187}{432}$$

Now using the result from the paper on solving cubic equations we can write a mostly symbolic answer for w.

$$w = \sqrt[3]{\tfrac{187}{432} + \sqrt{\left(\tfrac{187}{432}\right)^2 - \tfrac{343}{729}}} + \sqrt[3]{\tfrac{187}{432} - \sqrt{\left(\tfrac{187}{432}\right)^3 - \tfrac{343}{729}}}$$

We then have $z = w - \frac{5}{3}$.

$$z = \sqrt[3]{\tfrac{187}{432} + \sqrt{\left(\tfrac{187}{432}\right)^2 - \tfrac{343}{729}}} + \sqrt[3]{\tfrac{187}{432} - \sqrt{\left(\tfrac{187}{432}\right)^2 - \tfrac{343}{729}}} - \tfrac{5}{3}.$$

Although the square roots involve complex numbers, when the two cube roots are added the imaginary parts cancel and we can arrive at real solutions. Using a computer, we can find three different real decimal approximations for w.

$$w \approx -1.28908499330282 \qquad w \approx -0.398063972592354 \qquad w \approx 1.68714896216989$$

These answers lead to three approximations for z:

$z \approx -2.95575165997$ or $z \approx -2.06473063926$ or $z \approx 0.0204822955$

Next we factor the quadratic as a perfect square.

$$\left\{\sqrt{6+2z}\,y - \sqrt{z^2+2z}\right\}^2 = (6+2z)y^2 - 2\sqrt{6+2z} \cdot y\sqrt{z^2+2z} + z^2 + 2z$$

$$= (6+2z)y^2 - 2\sqrt{(6+2z)(z^2+2z)} \cdot y + z^2 + 2z$$

$$= (6+2z)y^2 - \sqrt{4} \cdot \sqrt{2z^3 + 10z^2 + 12z} \cdot y + z^2 + 2z$$

$$= (6+2z)y^2 + \sqrt{8z^3 + 40z^2 + 48z} \cdot y + z^2 + 2z$$

$= (6+2z)y^2 + \sqrt{1} \cdot y + z^2 + 2z$

$= (6+2z)y^2 + y + z^2 + 2z$

Then $y^2 + 1 + z = \pm\left\{\sqrt{6+2z}\, y - \sqrt{z^2+2z}\right\}$

If we take the $+$ sign then

$$y^2 - \sqrt{6+2z}\, y + \left[1 + z + \sqrt{z^2+2z}\right] = 0$$

If we take the $-$ sign then

$$y^2 + \sqrt{6+2z}\, y + \left[1 + z - \sqrt{z^2+2z}\right] = 0$$

Because these last equations are quadratics in y we can solve both of them using the good old quadratic formula.

$$y = \frac{\sqrt{6+2z} \pm \sqrt{6+2z-4\left[1+z+\sqrt{z^2+2z}\right]}}{2}$$

or

$$y = \frac{-\sqrt{6+2z} \pm \sqrt{6+2z-4\left[1+z-\sqrt{z^2+2z}\right]}}{2}$$

This example demonstrates just how difficult it is to write an explicit expression for the answer for y, especially when z is so complicated. Since we found three approximations for z, we should check that none of the radicals involving y contain negative numbers. We can only use z-values that make all the radicals appropriate. For this example, we compute the following table$s$.

z	$6+2z$	z^2+2z
-2.95575165997	0.8849668006	2.82496455548
-2.06473063926	1.87053872148	0.13365133418
0.0204822955	6.040964591	0.04138411543

z	$6+2z-4\left[1+z+\sqrt{z^2+2z}\right]$	$6+2z-4\left[1+z-\sqrt{z^2+2z}\right]$
-2.95575165997	1.1884509709	14.634555669
-2.06473063926	4.66712706676	7.59179549028
0.0204822955	1.14531196284	2.77275885518

Finally we show the final four possible answers for y:

z	$y_1 +$	$y_2 -$	$y_3 +$	$y_4 -$
-2.95575165997	0.6938224584	-0.39633852988	1.76401492581	-2.06149885058
-2.06473063926	1.76401492581	-0.39633852988	0.6938224584	-2.06149885058
0.0204822955	1.76401492581	0.6938224584	-0.39633852988	-2.06149885058

Of interest here is that all three values of z lead to the same set of answers for y.

Finally, if we want to write the answers for x then we can simply subtract 1 from y and here is the final set of four answers to the equation $x^4 + 4x^3 + 2x^2 - 3x - 1 = 0$.

$$x \approx -3.06149885058 \quad \text{or} \quad x \approx -1.39633852988$$

$$x \approx -0.30617754161 \quad \text{or} \quad x \approx 0.76401492581$$

We haven't completed the original case where we assumed $P < 0$. Technically we should say what to do in the case where $P > 0$, but we will leave that to reader!

After reading this paper you may wish to read another paper that explains how the pure quartic can intersect any general parabola. That paper is titled **The Pure Quartic and Intersections With Quadratic Polynomials.** Just visit **http://homepage.smc.edu/kennedy_john**