The functional 7.1 can be computed in either coordinate system and, for reasons that will soon become apparent, we consider the integral in the $\overline{O}\overline{x}\,\overline{y}$ representation over a *limited*, but arbitrary, range

$$G = \int_{\overline{c}}^{\overline{d}} du\, F\left(\overline{y}(u), \overline{y}'(u)\right) \quad \text{where} \quad \overline{y}' = \frac{d\overline{y}}{du},$$

$\overline{c} = c - \delta$, $\overline{d} = d - \delta$ and where $a < c < d < b$. The integrand of G depends on u only through the function $\overline{y}(u)$: this means that at each value of u the integrand has the same value as the integrand of $S[y]$ at the equivalent point, $x = u + \delta$. Hence

$$\int_{\overline{c}}^{\overline{d}} du\, F\left(\overline{y}(u), \overline{y}'(u)\right) = \int_c^d dx\, F\left(y(x), y'(x)\right) \quad \text{where} \quad x = u + \delta, \tag{7.2}$$

and this is true for *all* δ.

Now consider small values of δ and expand to $O(\delta)$, first writing the integral in the form

$$\begin{aligned} G &= \int_{c-\delta}^{d-\delta} du\, F(\overline{y}(u), \overline{y}'(u)) \\ &= \int_c^d du\, F(\overline{y}, \overline{y}') + \int_{c-\delta}^{c} du\, F(\overline{y}, \overline{y}') - \int_{d-\delta}^{d} du\, F(\overline{y}, \overline{y}'). \end{aligned} \tag{7.3}$$

But, for small δ

$$\int_{z-\delta}^{z} du\, g(u) = g(z)\delta + O(\delta^2),$$

and to this order

$$\overline{y}(u) = y(u+\delta) = y(u) + y'(u)\delta + O(\delta^2), \quad \text{and} \quad \overline{y}'(u) = y'(u) + y''(u)\delta + O(\delta^2).$$

Thus the expression 7.3 for G becomes, to first order in δ,

$$\begin{aligned} G &= \int_c^d du\, F(y + y'\delta, y' + y''\delta) - \delta \Big[F(y, y')\Big]_c^d + O(\delta^2) \\ &= \int_c^d du\, \left\{F(y, y') + \delta \frac{\partial F}{\partial y} y' + \delta \frac{\partial F}{\partial y'} y''\right\} - \delta \Big[F(y, y')\Big]_c^d + O(\delta^2). \end{aligned}$$

Because of equation 7.2 this gives

$$0 = \delta \int_c^d du\, \left(y' \frac{\partial F}{\partial y} + y'' \frac{\partial F}{\partial y'}\right) - \delta \Big[F(y, y')\Big]_c^d + O(\delta^2). \tag{7.4}$$

Now integrate the second integral by parts,

$$\int_c^d du\, y'' \frac{\partial F}{\partial y'} = \left[y' \frac{\partial F}{\partial y'}\right]_c^d - \int_c^d du\, y' \frac{d}{du}\left(\frac{\partial F}{\partial y'}\right).$$

Substituting this into 7.4 and dividing by δ gives

$$0 = \left[y' \frac{\partial F}{\partial y'} - F\right]_c^d - \int_c^d du\, y' \left\{\frac{d}{du}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y}\right\} + O(\delta). \tag{7.5}$$