Introduction to Spacetime Geometry

Let's start with a review of a basic feature of Euclidean geometry, the Pythagorean theorem. In a two-dimensional coordinate system we can relate the length of a line segment to the coordinates of its endpoints using the following relation.

$$(\Delta s)^2 = (\Delta x)^2 + (\Delta y)^2$$

For example, the length of the line segment shown in Figure 1 is 10 m.

$$(\Delta s)^2 = (\Delta x)^2 + (\Delta y)^2 = (7\,\mathrm{m} - 1\,\mathrm{m})^2 + (10\,\mathrm{m} - 2\,\mathrm{m})^2 = (6\,\mathrm{m})^2 + (8\,\mathrm{m})^2 = (10\,\mathrm{m})^2$$

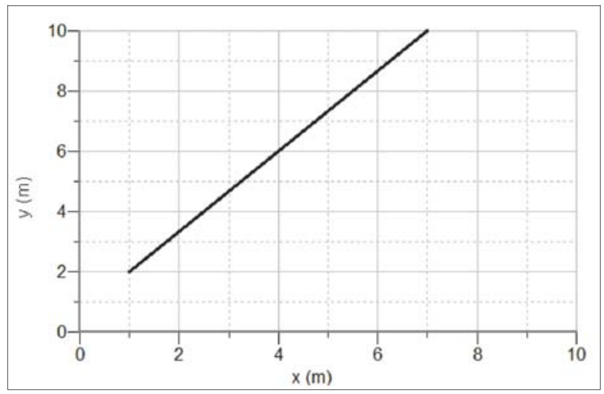

Fig 1: A line segment of length 10 m.

One of the properties of the length of a line segment is that it's the same regardless of the coordinate system used to measure it, something that would be difficult to demonstrate if we measured *x* and *y* in different units.

In spacetime geometry there's an analogous relationship:

$$(\Delta t_o)^2 = (\Delta t)^2 - (\Delta x)^2$$

where *t* is time, *x* is position, and Δt_o is called the proper time. But position and time must be measured in the same units, such as minutes, where a minute of distance is how far light travels in a time of one minute. For example, light takes 8 minutes to travel from the sun to Earth, so the distance is 8 minutes. Note that the speed of light *c* equals one minute per minute. This equation is valid only in systems of units like this, where $c = 1$.

Using these units, let's look at an example. Suppose we have a rocket that moves along the *x*-axis. We are interested in two events, the first event is when the rocket has a position of 3, and then later when it has a position of 6. We also note that the first event occurs at a time of 2 and the second occurs at a time of 7.

We calculate the speed of the rocket: $\beta = \dfrac{\Delta x}{\Delta t} = \dfrac{(6-3)}{(7-2)} = \dfrac{3}{5} = 0.6$

And the proper time is $\Delta t_o = \sqrt{(\Delta t)^2 - (\Delta x)^2} = \sqrt{(7-2)^2 - (6-3)^2} = \sqrt{(5)^2 - (3)^2} = 4\,\text{minutes}.$

This means that only 4 minutes of time passes for people aboard the rocket!

The result of a calculation like this is so unusual that it causes us to scratch our heads and wonder how we can understand it. We have to use care in explaining what's happening. The axis we used to measure the rocket's position is at rest relative to us. Likewise the clock we used to measure time is also at rest relative to us. They form what we call our frame of reference. Note that in our frame of reference we are at rest.

Figure 2 shows such a frame of reference. We imagine an object such as a rocket moving along the *x*-axis. When the object is located at some particular position we call it an event. We use the clock to measure the time *t* of the event and the x-axis to measure the position *x* of the event. We say that *x* and *t* are the coordinates of the event.

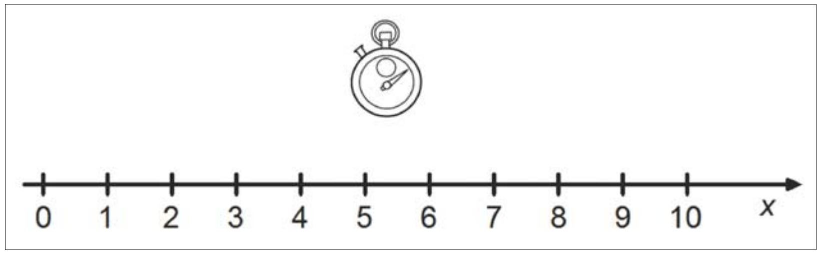

Fig 2: Viewing a frame of reference used to measure position *x* and time *t*. The viewer is at rest relative to the frame of reference.

The example we just looked at involved two events. In our frame of reference the first event occurred at position $x = 3$ and time $t = 2$, where we are measuring position in minutes and time in minutes. The second event occurred at $x = 6$ and $t = 7$.

Shown in Figure 3 is a spacetime diagram of this situation. To make clear that spacetime geometry is not like Euclidean geometry, let's take another look at what's being shown in Figure 3.

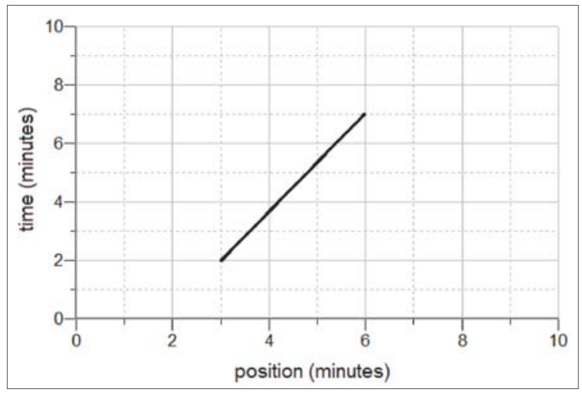

Fig 3: A proper time of 4 minutes.

Figure 4 shows the triangle we're using to do our spacetime calculation. Note that it's a strange looking triangle because the hypotenuse is not the longest side! The type of thinking that we're used to doing just won't help us out here. The geometry is not Euclidean.

We do not use the Pythagorean theorem and Euclidean geometry. We use the definition of proper time and spacetime geometry.

$$(\Delta t_o)^2 = (\Delta t)^2 - (\Delta x)^2$$

$$(4)^2 = (5)^2 - (3)^2$$

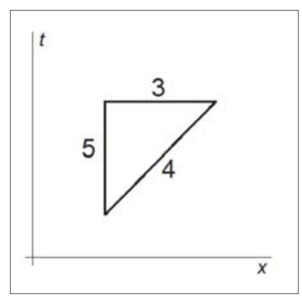

Fig 4: A spacetime triangle.

Note that people aboard the rocket have their frame of reference. It's a frame of reference in which they are at rest. It's the clock in their frame of reference that reads an elapsed time of 4 minutes between the events.

One of the things that makes this seem strange is that we are used to looking at things from our own frame of reference, and thinking that what we measure would be the same when measured in every frame of reference. We could instead think of our planet Earth as our rocket, traveling through space at a speed of $0.6c$ relative to some observer. Suppose two events occur at the same position, and are separated by 4 minutes of time, in our frame of reference. Those same two events are separated by 5 minutes of time in that observer's frame of reference.

Is there another way of relating these clock readings that doesn't involve geometry? Yes, it's the familiar formula for time dilation.

$$\Delta t = \gamma\,\Delta t_o$$

In our example we have $\beta = 0.6$ and therefore

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} = \frac{1}{\sqrt{1-0.6^2}} = 1.25$$

So the time in our frame of reference is

$$\Delta t = \gamma\,\Delta t_o = (1.25)(4) = 5$$

Muon Decay: A Worked Example

Muons are particles that were first discovered in 1936. The muon is a much heavier cousin of the electron, and it's unstable with a mean lifetime of 2.2 μs. Some will live for a longer time, some shorter, but on average, they live for 2.2 μs before decaying. Muons are created in Earth's upper atmosphere, and some of them subsequently travel toward Earth's surface. For this example we'll suppose we have a single muon moving toward Earth's surface at a typical speed of 0.9997*c*, and that it lives for 2.2 μs before decaying.

To calculate the lifetime of this muon in our frame of reference we first calculate the value of γ.

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} = \frac{1}{\sqrt{1-0.9997^2}} \approx 40.828$$

Giving us a lifetime of

$$\Delta t = \gamma\,\Delta t_o = (40.828)(2.2\,\mu\text{s}) \approx 89.821\,\mu\text{s}.$$

Therefore the distance it travels in our frame of reference is

$$\Delta x = v\Delta t = (0.9997)\left(3\times10^8\ {}^{\text{m}}\!/\!_{\text{s}}\right)\left(89.821\times10^{-6}\ \text{s}\right) \approx 27\,000\,\text{m}.$$

If we had instead used the proper time of 2.2 μs to do this calculation we'd have a distance traveled of only 660 meters. Such a difference is so huge that it's easy to detect experimentally. Earth's atmosphere reaches to a height of more than 30 000 meters, so if the average muon decayed after traveling only 660 meters we'd expect very few to make it to Earth's surface. About 90 years has passed since the discovery of muons in our atmosphere, but even then physicists knew that a large fraction of the muons created near the top of the atmosphere would make it to Earth's surface. This is just one of many experimental results and measurements that confirm the validity of Einstein's relativity.

Let's look at the spacetime geometry. Remember that we need a system of units where *c* = 1, so we shall choose microseconds for units of time and microseconds for units of distance. A microsecond of distance, the distance light travels in one microsecond, is $(3\times10^8\ {}^{\text{m}}\!/\!_{\text{s}})(1\times10^{-6}\ \text{s}) = 300\,\text{m}$.

First, we calculate that $\Delta x = \beta \Delta t = (0.9997)(89.821) = 89.794.$

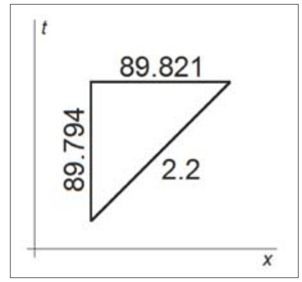

Fig 5: Spacetime diagram for flight of muon.

Finally, we verify that the definition of proper time is satisfied:

$$(\Delta t_o)^2 = (\Delta t)^2 - (\Delta x)^2$$

$$(2.2)^2 \approx (89.821)^2 - (89.794)^2$$

Figure 5 shows the spacetime diagram for the flight of the muon. Note that the two legs of the right triangle are almost identical in length (because the muon's speed is very near the speed of light) and that again the hypotenuse is not the longest side.

The Twin Paradox: A Worked Example

Two twins, Tess and Sam, test out the validity of relativity by doing an experiment. Sam stays at home while Tess travels. Tess spends one hour traveling at a speed of 0.6*c*, then turns around and heads back home at the same speed. When she gets back 2 hours have elapsed on her clock, but 2.5 hours have elapsed on Sam's clock. Scratching their heads, they try to use the definition of proper time to figure out the discrepancy in their clock readings.

To do their spacetime calculations they decide to measure time in minutes and distance in minutes. A minute of distance is how far light travels in a minute of time: $(3\times 10^8\ {}^{\mathrm{m}}\!/\!_{\mathrm{s}})(60\,\mathrm{s}) = 1.8\times 10^{10}\ \mathrm{m}.$

First, we look at things from Sam's frame of reference. He imagines an *x*-axis with the origin at his house on planet Earth. It stretches out into space for several minutes of distance. Each minute representing the distance a light beam would travel in one minute of time.

Sam realizes that half of 2.5 hours is 75 minutes of time. He uses this to calculate the distance to the turn-around point:

$$\Delta x = \beta\, \Delta t = (0.6)(75) = 45.$$

So there are two events, Event 1 is when Tess leaves, and Event 2 is when Tess reaches the turn-around point. In Sam's frame of reference they occur at *x* = 1 and *x* = 45, respectively.

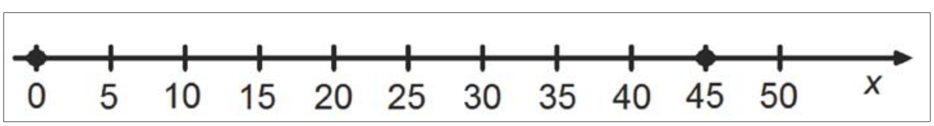

Fig 6: Event 1 is the departure, occurs at *x* = 1. Event 2 is the turn-around, occurs at *x* = 45.

Now let's look at things from Tess's frame of reference. These two events occur at the same place in her frame of reference, so the amount of time that elapses between them is a proper time. The relationship between her coordinates and his is given by

$$\Delta t_o = \sqrt{(\Delta t)^2 - (\Delta x)^2} = \sqrt{(75)^2 - (45)^2} = 60.$$

Figure 7 is the spacetime diagram Sam drew of the outbound half of Tess's trip. We can verify the calculation of the elapsed time using the time dilation formula. Recalling that γ = 1.25 when β = 0.6.

$$\Delta t = \gamma\,\Delta t_o = (1.25)(60) = 75.$$

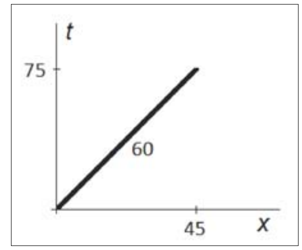

Fig 7: Sam's spacetime diagram of Tess's outbound trip.

So, from Sam's perspective Tess takes 60 minutes to go out. And then she would take 60 minutes to come back, for a total time of 2 hours. But for Sam it's 75 minutes of waiting for Tess to go out, 75 minutes for her return, for a total of 2.5 hours!

The question arises, and this is the apparent paradox, why can't we look at things from Tess's frame of reference, and conclude the reverse?

In other words, Tess imagines an *x*-axis stretching out from her space ship. In her frame of reference she stays at the origin, but she sees Sam as moving away from her along that *x*-axis until he gets to the turn-around point. Then it would be her that experiences 75 minutes of time while he experiences 60. In fact, that is a perfectly valid way to look at it! Figure 6 could just as well be Tess's spacetime diagram of Sam's outbound journey. It seems strange to think of Sam as going anywhere when he stays at home, but from Tess's point of view that's exactly what she sees Sam doing.

You could imagine a trip like this occurring at a much faster speed and for a much longer time, so the difference is more pronounced. The traveling twin could be gone for 50 years, and be only 5 years older upon return. The stay-at-home twin is 50 years older and is now a grandfather. It seems perfectly reasonable to ask why the reverse doesn't happen so that the traveling twin is 50 years older and the stay-at-home twin is only 5 years older.

The resolution of the paradox lies in looking at the complete spacetime diagram of both the outbound and the return legs of the journey, as shown in Figure 9. This has to be a diagram of Tess's journey from Sam's perspective, it cannot be a diagram of Sam's journey. The reason is the kink at the half-way point where Tess turns around and heads back home. Such a change in direction is something that only Tess experiences. Even if she had her eyes closed she would notice the acceleration from turning around. She can feel it. Sam experiences no such thing at the half-way point. He could be relaxing in a chair, eyes closed or open, and he'd feel nothing. No acceleration. It's not he who turns around, it's she. That's what breaks the symmetry and lets us realize that the scenario where the stay-at-home twin ages more is the correct one.

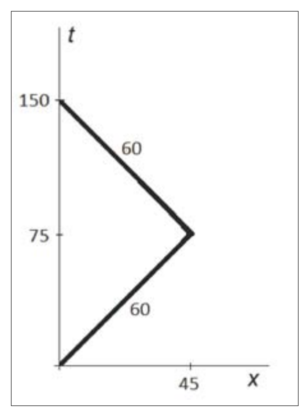

Fig 9: Sam's spacetime diagram of Tess's complete trip.

A more in-depth analysis is required to work things out from Tess's two different frames of reference, the one she's in during the outbound half, and the one she switches to for the return half. Doing one of these more complete analyses gives of course the same result. Tess experiences a proper time of 2 hours, Sam experiences a proper time of 2.5 hours. The difference in their ages is a difference in proper times. Note that this is not the same thing as the difference between a proper time and a dilated time. A difference in proper times is something all observers will agree on, whereas the difference between an observed dilated time and a proper time depends on the observer's frame of reference.

Exercises

1. A subatomic particle lives for 7 microseconds before decaying, as measured in its own frame of reference.

(a) Calculate its lifetime as measured by an observer with a relative speed of 0.96*c*.

(b) Sketch a spacetime diagram of the particle's motion from the frame of reference of the observer. Draw the diagram to scale. Label the axes with numbers. Label the length of the lines drawn on the diagram.

2. One twin leaves home and travels in a straight line at a speed of 0.96*c* for 7 years, as measured in his own frame of reference. He then turns around and comes back home at the same speed, having aged 14 years during the trip.

(a) His twin sister, who stayed at home, ages how much during her brother's absence?

(b) Sketch a spacetime diagram of the traveling twin's motion from the frame of reference of the stay-at-home twin. Draw the diagram to scale. Label the axes with numbers. Label the lengths of the lines drawn on the diagram.