§III.3.7 Twistors, Spinors and the Einstein Vacuum Equations

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Abstract. A link is established between twistor theory and the structure of Einstein's equations.

Whilst studying the twistor theory of hypersurfaces in space-time, the author discovered a symmetric covariant tensor of rank two on a three-sphere bundle over space-time, which, when restricted to a hypersurface gave the Fefferman conformal structure of the twistor C.R. structure of that hypersurface (see below, for definitions) (Sparling 1982, §III.3.6). Whilst writing up that work, the author suddenly realized that the symmetric tensor was but the symmetric part of a naturally defined tensor of rank two, whose skew part controls the Einstein equations and explains the origin of the Witten argument for positive energy (Witten 1981). The tensor is analogous to the Kähler tensor of Kählerian complex manifold, whose symmetric part gives the Kähler metric and whose skew part gives the Kähler symplectic structure. A discussion of its definition and basic properties follows.

Let M be a smooth four-manifold with smooth Lorentzian metric g. Over M, form the principal bundle K of orthonormal frames of M, δ_i, where $i = 0, 1, 2, 3$. Take δ_0 to be null and the δ_i to have scalar products with g, $g_{ij} \equiv g(\delta_i, \delta_j)$; $g_{03} = g_{30} = -g_{11} = -g_{22} = 1$; $g_{ij} = 0$ otherwise. Indices are lowered with g_{ij}, raised with its inverse g^{ij} and the Einstein summation convention is used.

On K, construct the canonical form θ^i and a connection form $\theta_i{}^j = -\theta^j{}_i$. θ^i links the abstract bundle K with the tangent geometry of M. $\theta_i{}^j$ represents a metric preserving connection for M (Kobayashi and Nomizu 1963). The Lie algebra of the Lorentz group acts on K, giving six vertical vector fields, $D_i{}^j = -D^j{}_i$. Denote by D_i the four horizontal connection vector fields. Then θ^i, $\theta_i{}^j$, $D_i{}^j$ and D_i obey the relations:

$$\langle \theta^i, D_j \rangle = \delta^i_j, \langle \theta^i, D_k{}^m \rangle = 0, \tag{1}$$

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$$\langle \theta_i{}^j, D_m \rangle = 0, \langle \theta_i{}^j, D_m^\ell \rangle = \frac{1}{2}(\delta_i{}^\ell \delta_m^j - g_{im} g^{\ell j}), \tag{2}$$

$$d\theta^i + \theta_j{}^i \wedge \theta^j = \theta^k \wedge \theta^m R_{km}{}^i, \tag{3}$$

$$d\theta_j{}^i + \theta_k{}^i \wedge \theta_j{}^k = \theta^k \wedge \theta^m R_{kmj}{}^i, \tag{4}$$

$$[D_i{}^j, D_k^\ell] = \frac{1}{2}(-g_{ik} D^{j\ell} + \delta_i{}^\ell D^j{}_k - g^{j\ell} D_{ik} + \delta_k{}^j D_i{}^\ell), \tag{5}$$

$$[D_i{}^j, D_k] = \frac{1}{2}(\delta_k{}^j D_i - g_{ik} D^j), \tag{6}$$

$$[D_i, D_j] = -2R_{ijk}{}^\ell D_\ell{}^k - 2R_{ij}{}^k D_k. \tag{7}$$

Here $\langle \, , \rangle$ denotes the canonical pairing of a one form with a vector field, d is the exterior derivative, $\wedge$ the exterior product of forms and $[,]$ denotes the Lie bracket of vector fields. $\delta_i{}^j$ is the Kronecker delta. $R_{km}{}^i$ and $R_{kmj}{}^i$ represent, respectively, the torsion of curvature of the connection determined by $\theta_j{}^i$.

Define on K a second rank covariant tensor system S_i, which will be named the Fefferman tensor, given by:

$$S_i \equiv \frac{1}{2}\varepsilon_{ijkm}(\theta^j \otimes \theta^{km}), \tag{8}$$

where ε_{ijkm} is totally skew and $\varepsilon_{0123} = 1$. Denote by P_i the symmetric part of S_i and by L_i the skew part of S_i, so $S_i = P_i + L_i$ and L_i is the two-form:

$$L_i = \frac{1}{2}\varepsilon_{ijkm}(\theta^j \wedge \theta^{km}). \tag{9}$$

The goal of this work is to discuss P_i and L_i. First take the exterior derivative of L_i. One finds, from equations (3) and (4):

$$dL_i = \frac{1}{2}\varepsilon_{ijkm}[-\theta_n{}^j \wedge \theta^n \wedge^{km} + \theta^j \wedge \theta_n{}^m \wedge \theta^{kn} + \theta^p \wedge \theta^q \wedge^{km} R_{pq}{}^j - \theta^j \wedge \theta^p \wedge \theta^q R_{pq}{}^{km}]. \tag{10}$$

Now $\theta^j \wedge \theta^p \wedge \theta^q = \varepsilon^{jpqr}\Sigma_r, \Sigma_r \equiv \frac{1}{6}\varepsilon_{rstu}\theta^s \wedge \theta^t \wedge \theta^u$. So

$$\begin{aligned}
\varepsilon_{ijkm}\theta^j \wedge \theta^p \wedge \theta^q R_{pq}{}^{km} &= \varepsilon_{ijkm}\varepsilon^{jpqr}\Sigma_r R_{pq}{}^{km} \\
&= 2(\delta_i{}^p \delta_k{}^q \delta_m^r + \delta_i{}^r \delta_k{}^p \delta_m^q + \delta_i{}^q \delta_k{}^r \delta_m^p)\Sigma_r R_{pq}{}^{km} \\
&= 2(\Sigma_m R_{ik}{}^{km} + \Sigma_i R_{km}{}^{km} + \Sigma_k R_{mi}{}^{km}) \\
&= -4G_i{}^m \Sigma_m,
\end{aligned} \tag{11}$$

where $G_i{}^m \equiv R_{ik}{}^{mk} - \frac{1}{2}R_{jk}{}^{jk}\delta_i{}^m$ is, in the absence of torsion, the Einstein tensor. Define

$$E_i \equiv -\frac{1}{2}\varepsilon_{ijkm}\{\theta^j \wedge \theta^{kn} \wedge \theta_n{}^m - \theta^n \wedge \theta_n{}^j \wedge \theta^{km}\}. \tag{12}$$

Then (10) now reads, using (11),

$$dL_i = E_i + \frac{1}{2}\varepsilon_{ijkm}\theta^p \wedge \theta^q \wedge \theta^{km} R_{pq}{}^j + 2G_i{}^m \Sigma_m. \tag{13}$$

Einstein's vacuum equations for a connection preserving the metric are

$$R_{pq}{}^j = 0, \quad G_i{}^m = 0. \tag{14}$$

So from (13), one finds that E_i is closed (and exact), if Einstein's vacuum equations hold. This motivates one to calculate dE_i. From equations (12), (3) and (4) one finds that the terms of dE_i not explicitly involving the torsion or the curvature identically cancel. The residual terms give:

$$\begin{aligned} dE_i = -\frac{1}{2}\varepsilon_{ijkm}\{&\theta^p \wedge \theta^q \wedge R_{pq}{}^\ell[\delta_\ell{}^j\theta^{kn} \wedge \theta_n{}^m - \theta_\ell{}^j \wedge \theta^{km}] \\ &+\Sigma_r \wedge \varepsilon^{npqr} R_{pqn}{}^j\theta^{km}\} + 2\Sigma_k \wedge \theta_i{}^j G_j{}^k. \end{aligned} \tag{15}$$

From (15) one sees that if dE_i vanishes, then first $R_{pq}{}^j$ must vanish so $\varepsilon^{mpqr} R_{pqm}{}^j$ must vanish by a Bianchi identity, so the remaining term must vanish also: $G_i{}^r = 0$. This proves the theorem:

The Einstein vacuum equations hold if and only if E_i is closed. E_i is then exact and $E_i = dL_i$.

An homogeneous ideal $\mathcal{T}$ of forms, on a manifold X, is said to be differential if it is closed under exterior differentiation: $d\mathcal{T} \subset \mathcal{T}$. It is not difficult to show from equations (13) and (15) that the following are equivalent:

1) E_i generates a differential ideal;

2) E_i and S_i generate a differential ideal;

3) g is conformal to a metric satisfying Einstein's vacuum equations.

A tighter formulation is obtained if one passes from K to an associated bundle K/H with fibre $O(1,3)/H$ where H is a closed subgroup of $O(1,3)$. H will be connected, with Lie algebra h. The action of H on K gives a system of vector fields $\tilde{h}$ on K corresponding to h. Dividing out by the action of $\tilde{h}$ gives the associated bundle K/H. The aim is to produce geometrical quantities on K that pass down to K/H. For example, a form on K is the pullback of a form on K/H, if it is Lie derived by the action of H and its inner product with any vector of $\tilde{h}$ is zero. Consider the bundles $K_i \equiv K/H_i$, $i = 1, \ldots, 4$, where the $\tilde{h}_i$ are spanned by:

$$\tilde{h}_1 : D_{01}, D_{02}; \tilde{h}_2 : D_{01}, D_{02}, D_{12};$$

$$\tilde{h}_3 : D_{01}, D_{02}, D_{03}; \tilde{h}_4 : D_{01}, D_{02}, D_{12} \text{ and } D_{03}.$$

The fibres of the K_i at a point x of M are four copies of, respectively, $RP^3 \times R$, $S^2 \times R$, RP^3 and S^2. K_2 and K_4 are, respectively, the bundles of null vectors and null directions, regarded as either future or past pointing, with either of two orientations, accounting for the four copies. K_1 and K_3 are Hopf fibred over K_2 and K_4, respectively, with fibre the circle. The extra circular degree of freedom is that of (twice) the phase of a spinor that is ignored when the spinor is represented as a null vector. So K_1 and K_3 are spinorial in nature, K_2 and K_4 are not (when M has a spin structure, the bundle W of non-zero Weyl spinors maps $2:1$ to K_1, the spinor ψ and $-\psi$ mapping to the same point of K_1).

Consider now D_0, $S_0 = P_0 + L_0$ and E_0. D_0 passes down to K_1, K_2, and, up to proportionality, to K_3 and K_4. In each case it represents the null geodesic spray. S_0 passes down to K_1 and, up to scale, to K_3. E_0 passes down to K_1.

One now obtains:

Einstein's vacuum equations hold if and only if $dE_0 = 0$, if and only if $E_0 = dL_0$. g is conformal to a solution of the vacuum equations if and only if E_0 generates a differential ideal, if and only if E_0 and L_0 generate a differential ideal.

Being formulated on K_1, the vacuum equations may therefore be regarded as spinorial in nature. More generally one has in K_1, $dL_0 = E_0 + G_0$, where, in the case of vanishing torsion, G_0 is the form induced by $2G_0{}^m\Sigma_m$. If M has a spin structure, a section ψ of W induces naturally a section $\tilde{\psi}$ of K_1. Restricting the forms L_0, E_0 and G_0 to $\tilde{\psi}$ and pulling pack to M, L_0 becomes the two-form on M, used by Nester, whose integral at infinity gives the energy momentum vector of the space-time, for suitable boundary conditions on the space-time and on ψ (Nester 1981). E_0 and G_0 are then three forms on M. If the strong energy condition holds, integrals of G_0 over a spacelike hypersurface have a definite sign. If suitable conditions are imposed on ψ, the integrand

and integrals of E_0 have the same sign. For example, if on a spacelike hypersurface, ψ obeys the Weyl neutrino equation and has zero normal derivative, then one recovers Witten's version of the positive energy theorem (Witten 1981).

An integral manifold of $\mathcal{T}$ on X is a submanifold Y, of X, such that $\mathcal{T}$, restricted to Y, vanishes. In the analytic case, Cartan developed a general theory of differential ideals and their integral manifolds (Cartan 1952–55). Applying this theory to analytic space-times, the generic integral manifold of the generated by E_0 on K_1 is found to be four dimensional. Generically, the integral manifold the defines, and is defined by a solution ψ of a nonlinear spinor equation on M. Using standard Dirac spinor notation this equation is

$$\varepsilon^{abcd}(D_a\bar{\psi})\gamma_5\gamma_b(D_c\psi) = 0, \quad \gamma_5\psi = \mathrm{i}\,\psi. \tag{16}$$

Somewhat similar equations occur in reference (Harvey and Lawson 1982).

Finally, one may restrict K and K_i, a priori, to a spacelike hypersurface, S, giving N, N_i, respectively. Then, to formulate S_0, L_0, P_0, E_0 restricted to N_1, one requires only the intrinsic metric and extrinsic curvature of S. One has:

The constraint equations on S, for Einstein's vacuum equations, hold if and only if $d(E_0 \mid N_1) = 0$ where $E_0 \mid N_1$ is the restriction of E_0 to N_1.

On K_3, L_0 is defined up to proportionality and has D_0 as its only zero eigenvector. K_3 is seven dimensional. Restricting K_3 to S gives the six-dimensional manifold N_3. Since D_0 represents the null geodesic spray, D_0 sticks out of N_3 and $L_0 \mid N_3$ is non-degenerate, giving a symplectic two form on N_3, up to proportionality. Similarly $P_0 \mid N_3$ gives a non-degenerate conformal metric of signature zero, defined up to the same factor of proportionality as is $L_0 \mid N_3$.

It remains to interpret the symmetric part P_i of S_i. Actually a direct interpretation will be given only to P_0. To explain P_0, one needs the notion of a regular Cauchy-Riemann (C.R.) structure of hypersurface type (Chern & Moser 1974).

For a $(2n+1)$-dimensional manifold X, a C.R. structure of hypersurface type is given by a subbundle V, of the complexified tangent bundle of X, of dimension n, such that V contains no non-zero real vectors and such that the Lie bracket $[v, w]$, of any sections v and w of V, is in V. The C.R. structure is termed regular if the Hermitian form determined by $i[v, \bar{w}]$, as v and $\bar{w}$ (the complex conjugate of w) vary, computed modulo vectors of V and $\bar{V}$, is invertible. C.R. structures of hypersurface type model abstractly the structure naturally acquired by a hypersurface X from the complex structure of an ambient complex manifold. If, in local holomorphic co-ordinates, $\{z^i\}$, $i = 0, \ldots, n$, X has defining equation $r(z^i, \bar{z}^i) = 0$, where $dr \neq 0$, then V is spanned by those linear combinations $a^i \partial/\partial z^i$ tangent to X: $a^i \partial r/\partial z^i = 0$, when $r = 0$, and the Hermitian form is determined by the quantities $a^i \bar{a}^j \partial^2 r/\partial z^i \partial \bar{z}^j$, as a^i varies. The trivial model is given by the hyperquadric $Q : |z^1|^2 + \ldots + |z^{p+1}|^2 - |z^{p+2}|^2 - \ldots - |z^{p+q+1}|^2 - 1 = 0$ in C^{p+q+1} where the Hermitian form is invertible of signature $|p-q|$. More generally, one asks: how nearly does a given C.R. structure resemble the trivial model? Chern, Moser and Webster solved this problem: a principal $SU(p+1, q+1)/Z_{p+q+2}$ bundle with connection, over X may be constructed whose curvature Ω must vanish for X to be locally C.R. isomorphic to the trivial model (Chern & Moser 1974). Remarkably, Fefferman managed to simplify drastically their construction: he produced a circle bundle $\tilde{X}$ over X, together with a conformal structure F on $\tilde{X}$, whose Weyl curvature vanishes if and only if Ω vanishes (Fefferman 1976).

Recall that N_3 is a circle bundle over N_4. Also $P_0 \mid N_3$ gives N_3 a conformal structure. Denote by U the subbundle of the complexified tangent bundle of K_4 spanned by D_0, $D_1 + iD_2$ and $D_{31} + iD_{32}$, passed down to K_4. Denote by V the intersection of U with the complexified tangent bundle of N_4. Then one has:

V defines a regular C.R. structure of hypersurface type and signature zero, for N_4. $P_0 \mid N_3$ gives the Fefferman conformal structure of (N_4, V), provided $\theta_i{}^j$ is torsion free.

In the analytic case, N_4 may be embedded as a hypersurface in a three-complex dimensional manifold, PT, such that V is given by those holomorphic vectors of PT tangent to N_4.

PT is the hypersurface projective twistor space, defined by Penrose in reference (Penrose 1975).

Even in the non-analytic case, one terms the C.R. structure of N_4 the twistor structure, since many of the standard results of the analytic twistor theory still hold.

An essentially identical construction gives the C.R. structure of any hypersurface in space-time. The construction being conformally invariant, it may be applied to the asymptotic null infinity (scri) of asymptotically flat space-times. In each case the Fefferman conformal structure is given by a suitable restriction of P_0. For scri, it coincides with the restriction to null twistors of the Kähler metric of asymptotic twistor space (Ko, Newman and Penrose 1977). In suitable co-ordinates the metric encodes the Bondi shear; its Weyl curvature incorporates the gravitational radiation (Ko, Newman and Penrose 1977).

For a general spacelike hypersurface, the twistor C.R. structure encodes the conformal data of the hypersurface: the intrinsic conformal metric and the trace-free part of the extrinsic curvature of the hypersurface.

The conformal data represents data for conformally flat space-time if and only if the Weyl curvature of the Fefferman conformal structure vanishes.

This Weyl curvature generalizes York's conformal tensor, reducing to it, when the trace-free part of the extrinsic curvature vanishes (York 1972).

The work outlined here should provide the basis of a new attempt to quantize gravity, following the canonical approach. Wave functionals would be *holomorphic* functionals of the *whole* conformal data rather then functionals of just the intrinsic conformal three geometry. The concept of holomorphic is formulated by providing a complex structure for small fluctuations in the geometry, about a given geometry. Here the geometry is represented by a suitable first sheaf cohomology group. For geometries that are asymptotically flat, splitting theorems of the type given by Andreotti and Hill should suffice to split the deformations into 'positive' and 'negative' frequency parts (Andreotti and Hill 1972). The complex structure is then given by multiplying the positive parts by i, the negative by $-i$. In flat space, on any spacelike Cauchy hypersurface, this procedure coincides with the usual quantization of spin two zero rest mass fields. For less global geometries, there would remain some ambiguous deformations, which could neither be classified as positive nor negative frequency. To handle these properly one might have to bring in concepts of intuitionistic logic — the ambiguous deformations corresponding to a non-trivial 'excluded middle'. At the quantum level one would appear to need a generalization of Fock space, including states that intrinsically are not combinations of particles and antiparticles alone. Physically, they seem to represent deformations with a length scale larger than the geometry under study. Particle detectors, which are expressly designed to remove all ambiguities, seem to correspond to logical morphisms taking intuitionistic to classical logic (Johnstone 1977).

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References

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