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Twistors, Spinors and the Einstein Vacuum Equations

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Abstract

A link is established between twistor theory and the structure of Einstein's equations.

Whilst investigating the twistor theory of hypersurfaces in spacetime, the author discovered a symmetric covariant tensor of rank two on a three sphere bundle over spacetime, which, when restricted to a hypersurface gave the Fefferman conformal metric of the twistor C.R. structure of that hypersurface (see below, for definitions).[1] Whilst writing this work up for this year's gravity prize essay contest, the author suddenly realized that the symmetric tensor was but the symmetric part of a naturally defined tensor of rank two, whose skew part controls the Einstein equations and explains the origin of the Witten argument for positive energy.[2] The tensor is similar in nature to the Kähler tensor of Kählerian complex manifold, whose symmetric part gives the Kähler metric and whose skew part gives the Kähler symplectic structure. Such a tensor may be regarded as defining an orthosymplectic structure for the manifold. A discussion of its definition and basic properties follows.

Let M be a smooth four manifold with smooth Lorentzian metric g. Over M, form the principal bundle K of orthonormal frames of M, δ_i, where $i = 0, 1, 2, 3$. Take δ_0 to be null and the δ_i to have scalar products with g, $g_{ij} \equiv g(\delta_i, \delta_j)$; $g_{03} = g_{30} = - g_{11} = - g_{22} = 1$; $g_{ij} = 0$ otherwise. On K, construct the canonical form θ^i and a connection form $\theta^{ij} = - \theta^{ji}$. θ^i links the abstract bundle K with the tangent geometry of M. θ^{ij} represents a metric preserving connection for M.[3] The Lie algebra of the Lorentz group acts on K, giving six vertical vector fields, $D_{ij} = - D_{ji}$. Denote by D_i the four horizontal connection vector fields. Then θ^i, θ^{ij}, D_{ij} and D_i obey the relations:

$$<\theta^i, D_j> = \delta^i_j \ , \ <\theta^i, D_{km}> = 0 \qquad (1)$$

$$<\theta^{ij}, D_k> = 0, \ <\theta^{ij}, D_{km}> = \frac{1}{2} (\delta^i_k \delta^j_m - \delta^j_k \delta^i_m) \qquad (2)$$

$$d\theta^i = - 2\theta^j{}_\wedge \theta_j{}^i - \frac{1}{2} \theta^k{}_\wedge \theta^m R_{km}{}^i \qquad (3)$$

$$d\theta_j{}^i = -2\,\theta_j{}^k \wedge \theta_k{}^i - \frac{1}{2}\,\theta^k \wedge \theta^m R_{kmj}{}^i \tag{4}$$

$$[D_{ij}, D_{km}] = g_{ik}D_{mj} - g_{im}D_{kj} - g_{jk}D_{mi} + g_{jm}D_{ki} \tag{5}$$

$$[D_{ij}, D_k] = -g_{ik}D_j + g_{jk}D_i \tag{6}$$

$$[D_i, D_j] = R_{ij}{}^{km}D_{km} + R_{ij}{}^k D_k \tag{7}$$

Here < , > denotes the canonical pairing of a one form with a vector field, d is the exterior derivative, $\wedge$ the exterior product of forms and [,] denotes the Lie bracket of vector fields. Indices are lowered with g_{ij}, raised with its inverse g^{ij} and the Einstein summation convention is used. δ^i_j is the Kronecker delta. $R_{km}{}^i$ and $R_{kmj}{}^i$ represent, respectively, the torsion and curvature of the connection determined by θ^{ij}.

Define on K a second rank covariant tensor system S_i, which will be called the Pauli-Lubanski spin tensor, given by:

$$S_i = \frac{1}{2}\,\varepsilon_{ijkm}\,(\theta^j \otimes \theta^{km}) \tag{8}$$

where ε_{ijkm} is totally skew and $\varepsilon_{0123} = 1$. Denote by P_i the symmetric part of S_i and by L_i the skew part of S_i, so $S_i = P_i + L_i$ and L_i is the two form:

$$L_i = \frac{1}{2}\,\varepsilon_{ijkm}\,(\theta^j \wedge \theta^{km}) \tag{9}$$

The goal of this work is to discuss P_i and L_i. First take the exterior derivative of L_i. One finds, from equations (3) and (4):

$$dL_i = \varepsilon_{ijkn}\,(-\theta^m \wedge \theta_m{}^j \wedge \theta^{kn} + \theta^j \wedge \theta^{km} \wedge \theta_m{}^n$$
$$- \frac{1}{4}\,\theta^p \wedge \theta^q \wedge \theta^{kn} R_{pq}{}^j + \frac{1}{4}\,\theta^j \wedge \theta^p \wedge \theta^q R_{pq}{}^{kn} \tag{10}$$

Let $\theta^j\wedge\theta^p\wedge\theta^q = \epsilon^{jpqr}\,\Sigma_r$, $\Sigma_r \equiv \frac{1}{6}\,\epsilon_{rstu}\,\theta^s\wedge\theta^t\wedge\theta^u$. So

$$\epsilon_{ijkm}\,\theta^j\wedge\theta^p\wedge\theta^q\,R_{pq}{}^{km} = \epsilon_{ijkm}\,\epsilon^{jpqr}\,\Sigma_r\,R_{pq}{}^{km}$$

$$= 2(\delta^p_i\,\delta^q_k\,\delta^r_m + \delta^r_i\,\delta^p_k\,\delta^q_m + \delta^q_i\,\delta^r_k\,\delta^p_m)\,\Sigma_r\,R_{pq}{}^{km}$$

$$= 2(\Sigma_m\,R_{ik}{}^{km} + \Sigma_i\,R_{km}{}^{km} + \Sigma_k\,R_{mi}{}^{km}) = -\,4\,G_i{}^m\,\Sigma_m \qquad (11)$$

where $G_i{}^m \equiv R_{ik}{}^{mk} - \frac{1}{2}\,R_{jk}{}^{jk}\,\delta_i{}^m$ is the Einstein tensor. Define

$$E_i \equiv \epsilon_{ijkn}\,\{\theta^j\wedge\theta^{km}\wedge\theta_m{}^n - \theta^m\wedge\theta_m{}^j\wedge\theta^{kn}\} \qquad (12)$$

Then (10) now reads.

$$dL_i = E_i - \frac{1}{4}\,\epsilon_{ijkm}\,\theta^p\wedge\theta^q\wedge\theta^{km}\,R_{pq}{}^j - G_i{}^m\,\Sigma_m \qquad (13)$$

Einstein's vacuum equations for a connection preserving the metric are

$$R_{pq}{}^j = 0,\ G_i{}^m = 0 \qquad (14)$$

So from (13), one finds E_i is closed (and exact) if Einstein's vacuum equations hold. This motivates one to calculate dE_i. From equations (3), (4) one finds that the terms of dE_i not explicitly involving the torsion or the curvature identically cancel. The residual terms give:

$$dE_i = -\,\frac{1}{2}\,\epsilon_{ijkr}\,\{\theta^p\wedge\theta^q\wedge R_{pq}{}^m\,(\delta^j_m\,\theta^{kn}\wedge\theta_n{}^r + \theta_m{}^j\wedge\theta^{kr})$$

$$-\,\Sigma_n\wedge\theta^{kn}\,(\epsilon^{mpqr}\,R_{pqm}{}^j)\} + 2\Sigma_r\wedge\theta_i{}^m\,G_m{}^r \qquad (15)$$

From (15) one sees that if dE_i vanishes, then first $R_{pq}{}^j$ must vanish, so $\epsilon^{mpqr}\,R_{pqm}{}^j$ must vanish by a Bianchi identity, so the remaining term must vanish also: $G_i{}^r = 0$. This proves the theorem:

E_i is closed if and only if the Einstein vacuum equations hold. E_i is then exact and $E_i = dL_i$.

A tighter formulation is obtained if one passes from K to an associated bundle K/H with fibre O(1,3)/H where H is a closed subgroup of O(1,3). H will be connected, with Lie algebra h. The action of H on K gives a system of vector fields $\tilde{h}$ on K corresponding to h. Dividing out by the action of $\tilde{h}$ gives the associated bundle K/H. The aim is to produce geometrical quantities on K that pass down to K/H. For example a form on K is the pullback of a form on K/H, if it is Lie derived by the action of H and its inner product with any vector of $\tilde{h}$ is zero. Consider the bundles $K_i = K/H_i$, $i = 1, \ldots, 4$, where the $\tilde{h}_i$ are spanned by:

$\tilde{h}_1$: D_{01}, D_{02}; $\tilde{h}_2$: D_{01}, D_{02}, D_{12}

$\tilde{h}_3$: D_{01}, D_{02}, D_{03}; $\tilde{h}_4$: D_{01}, D_{02}, D_{12} and D_{03}.

The fibres of the K_i at a point x of M are four copies of, respectively, $RP^3 \times R$, $S^2 \times R$, RP^3 and S^2. K_2 and K_4 are, respectively, the bundles of null vectors and null directions, regarded as either future or past pointing, with either of two orientations, accounting for the four copies. K_1 and K_3 are Hopf fibred over K_2 and K_4 with fibre the circle. The extra circular degree of freedom is that of (twice) the phase of a spinor that is ignored when the spinor is represented as a null vector. So K_1 and K_3 are spinorial in nature, K_2 and K_4 are not (when M has a spin structure, the bundle of Weyl spinors maps 2:1 to K_1, the spinor ψ and $-\psi$ mapping to the same point of K_1).

Consider now D_o, $S_o = P_o + L_o$ and E_o. D_o passes down to K_1, K_2, and, up to proportionality, to K_3 and K_4. In each case it represents the null geodesic spray. S_o passes down to K_1 and, up to scale, to K_3. E_o passes down to K_1. One now obtains: