

Sparse Channel Estimation via Matching Pursuit With Application to Equalization

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Abstract—Channels with a sparse impulse response arise in a number of communication applications. Exploiting the sparsity of the channel, we show how an estimate of the channel may be obtained using a **matching pursuit (MP) algorithm**. This estimate is compared to thresholded variants of the least squares (LS) channel estimate. Among these sparse channel estimates, the MP estimate is **computationally much simpler to implement and a shorter training sequence is required to form an accurate channel estimate leading to greater information throughput**.

Index Terms—Channel estimation, equalization, matching pursuit, sparsity.

I. INTRODUCTION

WE CONSIDER communication problems which involve the estimation and equalization of channels with a large delay spread but **with a small nonzero support**. Such channels, an example of which is given in Fig. 1, are described as having a sparse impulse response and are encountered in a number of different applications. For instance, in high-definition television (HDTV), there are few echoes but the channel response spans many hundreds of data symbols [1], [2]. In broad-band wireless communications, a “hilly terrain” (HT) delay profile consists of a sparsely distributed multipath [3]. **Underwater acoustic channels also exhibit a similar response** [4].

We propose exploiting the sparse nature of the channel through the use of a matching pursuit (MP) algorithm, which has been used in a wide variety of application areas [6], to derive an accurate estimate of the channel. This work expands upon initial results presented in [5]. **An estimate of the channel found using MP is inherently sparse in contrast to a least squares (LS) estimate of the channel wherein each of the tap values will in general be nonzero**. However, thresholding of the LS tap values (ThLS) may be used to provide a sparse channel estimate. Furthermore, an improved sparse LS estimate (SpLS) is obtained by reestimating the ThLS nonzero taps. We compare the accuracy of these channel estimates and quantify their effectiveness by considering their use in the equalization of sparse channels.

II. CHANNEL ESTIMATION VIA MATCHING PURSUIT (MP)

We first introduce some notation for the description of the communication system. A training sequence

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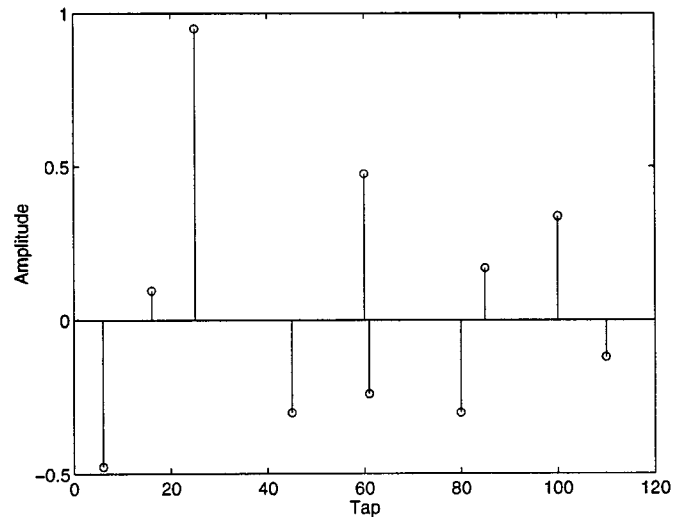


Fig. 1. Sparse channel with both precursor and postcursor components.

$s(m)$, $m = 0, \dots, M-1$ is transmitted through the channel and the received signal samples, $r(m)$, $m = 0, \dots, M-1$, can be expressed as

$$r(m) = \sum_{k=0}^{N-1} s(m-k)c(k) + n(m), \quad m = 0, \dots, M-1 \quad (1)$$

where the stationary channel impulse response is denoted by $c(m)$, $m = 0, \dots, N-1$. An example of a sparse channel, i.e., $c(m) \neq 0$ for very few values of m , is given in Fig. 1. The noise samples are given by $n(m)$, $m = 0, \dots, M-1$. We denote by σ_s^2 and σ_n^2 the signal and noise powers, respectively. The symbols $s(m)$, $m < 0$ may be obtained from previous decodings in a data stream [2] or can be assumed zero if this is the first packet received [3]. The received signal power is denoted by σ_r^2 . We can write (1) in matrix form as follows:

$$\begin{pmatrix} s(0) & s(-1) & \cdots & s(-N+1) \\ s(1) & s(0) & \cdots & s(-N+2) \\ \vdots & \vdots & & \vdots \\ s(M-1) & s(M-2) & \cdots & s(M-N) \end{pmatrix} \begin{pmatrix} c(0) \\ c(1) \\ \vdots \\ c(N-1) \end{pmatrix} = \begin{pmatrix} r(0) \\ r(1) \\ \vdots \\ r(M-1) \end{pmatrix} + \begin{pmatrix} n(0) \\ n(1) \\ \vdots \\ n(M-1) \end{pmatrix} \quad (2)$$

which can be abbreviated to

$$Ac = b + n. \quad (3)$$

TABLE I
SUMMARY OF CHANNEL ESTIMATES

LS	least squares estimate: $\hat{c}_{LS} = A^\dagger b$
ThLS	the LS estimate is thresholded by setting all taps to zero other than the L largest in magnitude, denoted $\{k_1, \dots, k_L\}$
SpLS	using the L taps obtained in ThLS, these taps are reestimated from (2) using the LS estimate $[a_{k_1}, \dots, a_{k_L}]^\dagger b$
OPT	the positions of the nonzero taps are assumed known (genie-aided) and the LS solution is found by employing only the corresponding columns in A
MP	derived using the method detailed in section 2.1

Estimation of a sparse channel was considered in an echo-cancellation context in [7] where the probing signal used was white noise or a first-order auto-regressive (AR) random process driven by white noise. A different estimation was proposed in [4] and more recently in [8] where again a white input was used. However, these methods cannot be generalized to the case of interest here where a sequence of symbols drawn from a finite alphabet is used in the training phase. Instead, we base our approach on an MP algorithm [9], [10].

From (3), since the channel is known to be sparse, the problem is to approximate the vector b in terms of a linear combination of a small number of columns from the matrix A , i.e., from (2) we must find c such that $Ac \approx b$. Thus, the problem can be viewed as a sparse representation problem and the MP algorithm described next gives an efficient method for obtaining a suboptimal solution to this problem.

A. Matching Pursuit

In the basic form of the MP algorithm [9], [10], we first find the column in the matrix $A = [a_1, a_2, \dots, a_N]$, which is best aligned with the signal vector $b_0 = b$ and this is denoted a_{k_1} . Then the projection of b_0 along this direction is removed from b_0 and the residual b_1 is obtained. Now the column in A , a_{k_2} , which is best aligned with b_1 is found and a new residual, b_2 , is formed. The algorithm proceeds by sequentially choosing the column which best matches the residual until some termination criterion is met. The p th iteration is described in the following paragraph.

We denote the projection onto a vector a_l as $P_{a_l} = a_l a_l^H / \|a_l\|^2$. The vector from A most closely aligned with the residual b_{p-1} is chosen, where the alignment is measured as the 2-norm (denoted by $\|\cdot\|$) of the projection of the residual onto the vector, i.e.,

$$k_p = \arg \max_l \|P_{a_l} b_{p-1}\| = \arg \max_l \frac{|a_l^H b_{p-1}|^2}{\|a_l\|^2},$$

$$l = 1, \dots, n, l \neq k_{p-1}. \quad (4)$$

The new residual vector is then computed as

$$b_p = b_{p-1} - P_{a_{k_p}} b_{p-1} = b_{p-1} - \frac{(a_{k_p}^H b_{p-1}) a_{k_p}}{\|a_{k_p}\|^2} \quad (5)$$

and the tap value at position k_p is $\hat{c}_{k_p} = (a_{k_p}^H b_{p-1}) / \|a_{k_p}\|^2$. The iteration is repeated until a specified number of taps, P , have been selected or the residual becomes sufficiently small, i.e., $\|b_p\| < \epsilon$.

In this algorithm, reselection of columns from A may occur but this has not been found to be a significant problem [11]. There are other variants of the MP algorithm [10] which can improve on the channel estimate produced here. However, the performance improvement is slight for this application and there is additional computational cost incurred [11].

III. CHANNEL ESTIMATION VIA LEAST SQUARES (LS)

The classical LS estimate of the channel, as used in [2] and [12], can be obtained from (2) as $\hat{c}_{LS} = A^\dagger b$ where $A^\dagger = (A^H A)^{-1} A^H$ is the pseudo-inverse of the matrix A . In general, each of the taps in this channel estimate will have a nonzero value and this will be shown to adversely affect the equalization performance. The number of nonzero taps may be reduced, so a sparse channel estimate is obtained, by thresholding the LS channel estimate. An upper bound, L , is placed on the number of nonzero taps in the channel and this allows us to form two thresholded estimates of the channel: thresholded LS (ThLS) and thresholded reestimated LS, referred to as sparse LS (SpLS). We also find the optimal genie-aided estimate of the channel (OPT). These channel estimates are detailed in Table I.

IV. CHANNEL ESTIMATE-BASED DFES

We consider the transmission of symbols through a sparse channel at a high data rate which results in intersymbol interference (ISI). An equalizer is required in the receiver to reduce this effect and ensure accurate decoding of the transmitted symbols. A decision feedback equalizer (DFE) is a popular choice as it gives good performance with moderate complexity [13], [14]. Through this application, we quantify the improvement obtainable by using an MP algorithm to estimate the channel over estimates based on LS.

The received signal samples are given by (1). The input to the slicer of a DFE [13], [15] is

$$\tilde{s}(m) = \sum_{k=-K}^0 f(k) r(m-k) + \sum_{k=1}^{N-1} w(k) \hat{s}(m-k). \quad (6)$$

The feedforward filter is of length $(K+1)$ and has taps $f(0), \dots, f(K)$. The feedback filter has length $(N-1)$ which corresponds to the memory of the channel and the taps are $w(1), \dots, w(N-1)$. The previously decoded symbols are $\hat{s}(m-1), \dots, \hat{s}(m-N+1)$. Based on a minimum mean squared error (MMSE) criterion at the slicer input, the optimum feedforward and feedback taps in a DFE are directly computable from an estimate of the channel [12], [15]. The

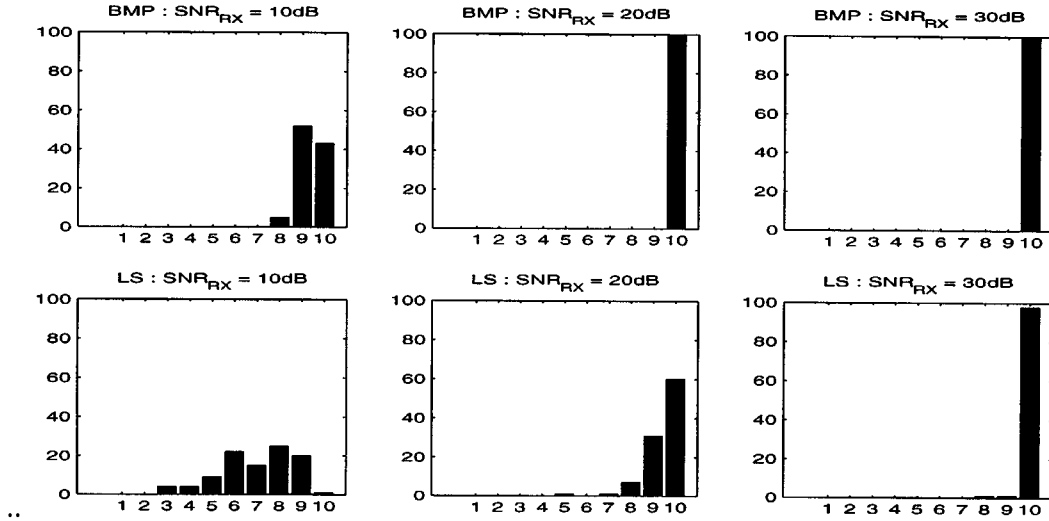


Fig. 2. Comparison of MP (top row) and Thresholded LS (ThLS) with the 20 largest magnitude taps chosen (bottom row). Each histogram gives the percentage of times that the selected nonzero set of taps contains 0–10 of the actual true nonzero taps as given in Fig. 1.

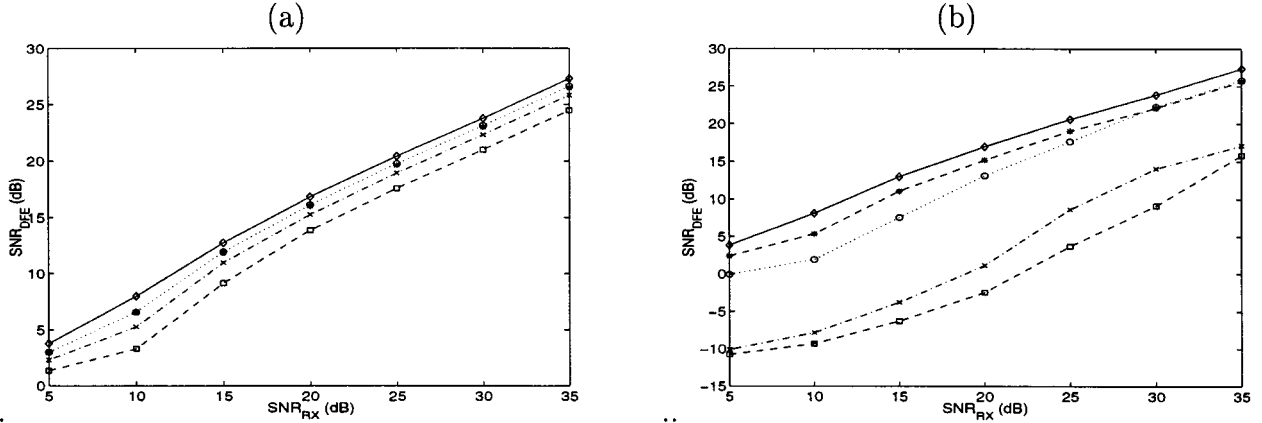


Fig. 3. Sequence lengths set to: (a) $M = 240$ and (b) $M = 148$. Plots of mean SNR_{DFE} values obtained using channel estimates LS(\square), ThLS(\times), SpLS(\circ), MP($*$), and OPT(\diamond) over a range of values of SNR_{RX} .

accuracy of the channel estimate is then critical to the equalization performance. The dependence of the bit error probability on the channel error covariance matrix has been considered in [14] where a Gauss–Markov model of the channel, along with a Kalman filter in the receiver, were employed. Here, we use simulations to quantify the effect of the channel estimate on the equalizer performance.

V. EQUALIZER SIMULATIONS

The channel to be equalized is given in Fig. 1. This represents a more complex channel than that considered in [2] and [5]. The channel length is set to $N = 119$. The length of the training sequence is M and we consider two values of M : $M = 148$, 240. This sequence is generated as a BPSK sequence where the symbols are equiprobable. The feedforward filter has $K = 79$ taps, as in [2], and the feedback filter has $N - 1 = 118$ taps.

Based on each of the channel estimates in Table I, the optimal DFE filters are obtained [15] and the equalization performance is compared. The slicer input SNR, SNR_{DFE} , is the metric used to compare the equalizers. This figure is obtained over a sequence of 10^6 transmitted symbols. A higher value of

SNR_{DFE} leads to lower bit error probability which is evaluated in [11] at lower values of receiver SNR, defined as $\text{SNR}_{\text{RX}} = 10 \log_{10}(\sigma_r^2/\sigma_n^2)$. One hundred trials are run with the training sequence varied in each trial and the results are then averaged. This is repeated for different values of SNR_{RX} .

A. Results

The channel in Fig. 1 has 10 nonzero taps. However, we assume that we do not have explicit knowledge of the channel and can only place an upper bound on the number of nonzero taps which is set to $L = 20$. The effect of varying the value of L is examined in [11]. Using a training sequence of length $M = 148$ bits, we give histograms in Fig. 2 of the number of nonzero taps correctly identified by the MP and ThLS channel estimates. This is repeated at different values of SNR_{RX} . From these histograms, it is clear that the MP gives a more consistent and more accurate channel estimate across a range of values of SNR_{RX} . At $\text{SNR}_{\text{RX}} = 10, 20$ dB, it is far superior to the ThLS estimate in correctly determining the nonzero taps. Another advantage of MP is that it is significantly cheaper to implement than a LS estimate provided the number of nonzero taps searched for is small [10].

The performance of the equalizers obtained from the channel estimates of Table I are now compared. In Fig. 3(a), results are plotted for a training sequence of length $M = 240$. It is seen that there is no distinguishable difference in performance (<0.1 dB) between the MP and SpLS based equalizers. The complexity of implementing the MP algorithm is much lower than that of the SpLS algorithm which requires the solution of two LS problems. The MP and SpLS algorithms give >1 dB improvement over the ThLS equalizer while this is in turn ≥ 1.5 dB better than the LS equalizer.

In Fig. 3(b), the training sequence has been shortened to $M = 148$ symbols. The reasons for considering a shorter training sequence are twofold. Firstly, the complexity in computing the channel estimates will be reduced in all cases since the inner products which need to be formed depend on M , as seen in (2). More importantly, if we can reduce the number of training bits required, this leads to an increase in throughput efficiency in systems where short packets are transmitted [3]. For instance, for a packet of length 2000 bits, reducing the training sequence length from 240 to 148 increases the efficiency from 88% to 92.6%. The ThLS and LS equalizers are now unusable. The MP and SpLS equalizers give essentially the same performance for $\text{SNR}_{RX} = 30, 35$ dB. However, it is seen that the performance of the MP equalizer degrades much more gracefully as SNR_{RX} is reduced. The performance is within 2 dB of the optimum for all levels of SNR_{RX} . This can be attributed to the greater accuracy of the MP channel estimate as seen in Fig. 2. In fact, the MP performance can be improved upon by reestimating the selected taps as done in obtaining SpLS but this is not implemented here as we wish to keep the complexity as low as possible. Comparing Fig. 3(a) and (b), the deterioration in MP based equalization performance incurred in reducing the length of the training sequence from $M = 240$ to $M = 148$ is small (<0.8 dB).

In summary, a low-complexity accurate estimate of the sparse channel can be obtained from a short training sequence through the use of an MP algorithm. This estimate is more robust to changes in SNR_{RX} than LS estimates based on this training sequence and is more computationally efficient to implement.

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