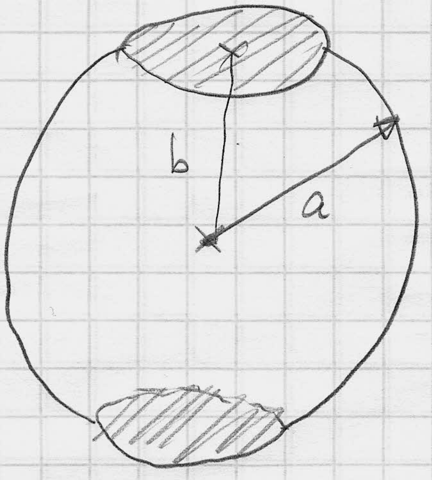
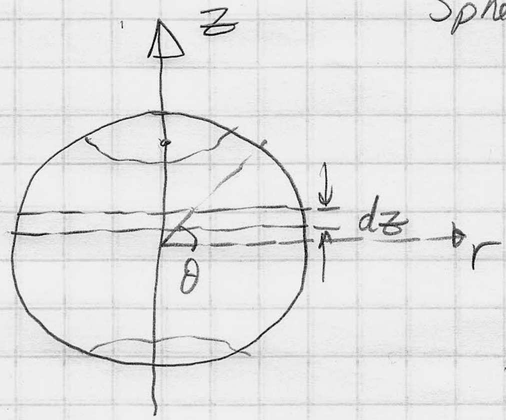
Resistance of a Truncated Sphere Claude Abraham



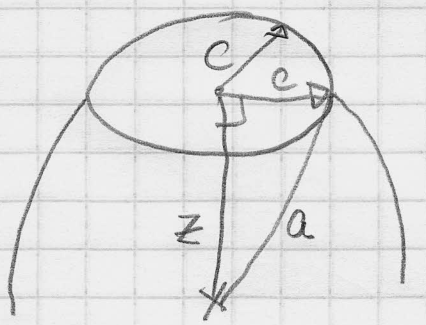
A sphere of radius a is truncated with a flat plane chord at radius b $(b<a)$, at both ends symmetrically, what is the resistance from end to end? Homogeneous sphere of resistivity ρ. The use of cylindrical coordinates is made.



Sphere equation in cylindrical system:

$r^2+z^2=a^2$, where r is the radial distance from the z axis.

Resistance is related to area, length, & resistivity for an incremental slab of material: $R=\rho\dfrac{l}{A}$, or $dR=\rho\dfrac{dl}{A}$, but $dl=dz$, & $A=A(z)=$ area of circle defined by plane transverse to z axis (normal). Radius of circle $\equiv c$ as follows



$c^2=a^2-z^2$, and $A(z)=\pi c^2$

$A(z)=\pi(a^2-z^2)$, so that

$$dR=\rho\frac{dl}{A(z)}=\rho\frac{dz}{\pi(a^2-z^2)}$$

To obtain R we integrate from $z=-b$ to $z=b$.

$$R = \int_{-b}^{b} \frac{\rho}{\pi} \frac{dz}{a^2 - z^2} = 2\frac{\rho}{\pi}\int_{0}^{b} \frac{dz}{a^2 - z^2}, \text{ due to symmetry.}$$

To evaluate the integral of $\frac{dz}{a^2 - z^2}$, we use partial fractions

$$\frac{1}{a^2 - z^2} = \frac{1}{(a-z)(a+z)} = \frac{c_1}{a-z} + \frac{c_2}{a+z}$$

$1 = c_1(a+z) + c_2(a-z)$ [after multiplying by $(a-z)(a+z)$]

$0 = (c_1 - c_2)z \Rightarrow c_1 = c_2$, $1 = c_1 a + c_2 a$ or $\frac{1}{a} = c_1 + c_2$

$\therefore c_1 = \frac{1}{2a} = c_2 \Rightarrow \frac{1}{a^2 - z^2} = \frac{1/2a}{a-z} + \frac{1/2a}{a+z}$

$$R = 2\frac{\rho}{\pi}\int_0^b \frac{dz}{a^2 - z^2} = 2\frac{\rho}{\pi} \cdot \frac{1}{2a}\int_0^b \left(\frac{1}{a-z} + \frac{1}{a+z}\right) dz$$

$$R = \frac{\rho}{\pi a}\Big[-\ln(a-z) + \ln(a+z)\Big]_0^b = \frac{\rho}{\pi a}\ln\left(\frac{a+z}{a-z}\right)\Big|_0^b$$

$$R = \frac{\rho}{\pi a}\left[\ln\left(\frac{a+b}{a-b}\right) - \ln\left(\frac{a+0}{a-0}\right)\right] = \frac{\rho}{\pi a}\left(\ln\left(\frac{a+b}{a-b}\right) - 0\right)$$

$$\boxed{R = \frac{\rho}{\pi a}\ln\left(\frac{a+b}{a-b}\right)}$$

As a check, if $b = a$, we have zero contact area (tangent point)

$R_{b=a} \Rightarrow \frac{\rho}{\pi a}\ln\left(\frac{2a}{0}\right) \Rightarrow \infty$, an open. If $b = 0$, we have non-zero area w/ zero length, $R = \frac{\rho}{\pi a}\ln\left(\frac{a}{a}\right) = 0$, a short.

let $b = \frac{a}{2}$, then $R = \frac{\rho}{\pi a}\ln\left(\frac{3a/2}{a/2}\right) = \frac{\rho}{\pi a}\ln 3$

or $R \approx 0.350\frac{\rho}{a}\,\Omega$. This value should be greater than the resistance of the outer cylinder, but less than that of the inner cylinder. With $b = 0.5a$, cross-section radius $c = \sqrt{a^2 - b^2} = \sqrt{a^2 - \frac{a^2}{4}} = a\frac{\sqrt{3}}{2}$

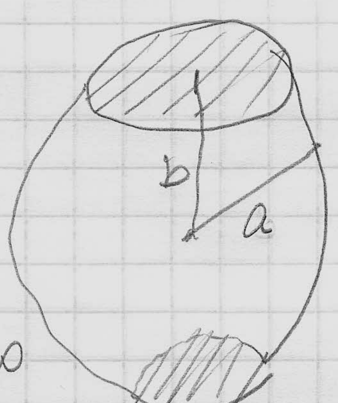
$\pi c^2 = \text{area} = \frac{3}{4}\pi a^2$. The cylinder length is $2b$ $= a$, so $R = \rho\frac{l}{A} = \rho\frac{a}{\frac{3}{4}\pi a^2} = \frac{4}{3\pi}\frac{\rho}{a} = 0.424\frac{\rho}{a}$.

The sphere section

resistance of $0.350\frac{\rho}{a}$ is indeed less than the inner cylinder resistance of $0.424\frac{\rho}{a}$. The outer cylinder has radius a, so that area $= \pi a^2$, so that $R = \rho\frac{l}{A}$

$R = \rho\frac{a}{\pi a^2} = \frac{1}{\pi}\frac{\rho}{a} = 0.318\frac{\rho}{a}$, which is less than

$0.350\frac{\rho}{a}$. We have agreement. The final answer



$$\boxed{R = \frac{\rho}{\pi a}\ln\left(\frac{a+b}{a-b}\right)}$$

Next problem: compute R if the terminating planes are <u>oblique</u>:

Much more difficult.

Claude Abraham 13 Dec 2012