$$\frac{\partial \Psi_{lm}^0}{\partial \phi} = \frac{\partial}{\partial \phi}\left(\sqrt{\left(\frac{2}{na}\right)^3 \frac{(n-l-1)!}{2n[(n+l)!]^3}} \exp\left(-\frac{r}{na}\right)\left(\frac{2r}{na}\right)^l L_{n-l-1}^{2l+1}\left(\frac{2r}{na}\right) Y_l^m(\theta,\phi)\right)$$

$$= \sqrt{\left(\frac{2}{na}\right)^3 \frac{(n-l-1)!}{2n[(n+l)!]^3}} \exp\left(-\frac{r}{na}\right)\left(\frac{2r}{na}\right)^l L_{n-l-1}^{2l+1}\left(\frac{2r}{na}\right)\frac{\partial}{\partial \phi}\left(Y_l^m(\theta,\phi)\right)$$

$$\frac{\partial}{\partial \phi}\left(Y_l^m(\theta,\phi)\right) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}e^{im\theta}\frac{\partial}{\partial \phi}P_l^m(cos(\phi))$$

$$= \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}e^{im\theta}(-sin(\phi))\frac{\partial}{\partial cos(\phi)}P_l^m(cos(\phi))$$

For positive m:

$$\frac{\partial}{\partial x}P_l^m(x) = \frac{\partial}{\partial x}\left((-1)^m(1-x^2)^{m/2}\frac{d^m}{dx^m}\frac{1}{2^l l!}\frac{d^l}{dx^l}\left(x^2-1\right)^l\right)$$

$$= \frac{m}{2}(-2x)(-1)^m(1-x^2)^{\frac{m}{2}-1}\frac{d^m}{dx^m}\frac{1}{2^l l!}\frac{d^l}{dx^l}\left(x^2-1\right)^l + (-1)^m(1-x^2)^{m/2}\frac{d^{m+1}}{dx^{m+1}}\frac{1}{2^l l!}\frac{d^l}{dx^l}\left(x^2-1\right)^l$$

$$= \frac{-mx}{1-x^2}(-1)^m(1-x^2)^{\frac{m}{2}}\frac{d^m}{dx^m}\frac{1}{2^l l!}\frac{d^l}{dx^l}\left(x^2-1\right)^l - \frac{1}{\sqrt{1-x^2}}(-1)^{m+1}(1-x^2)^{\frac{m+1}{2}}\frac{d^{m+1}}{dx^{m+1}}\frac{1}{2^l l!}\frac{d^l}{dx^l}\left(x^2-1\right)^l$$

$$= -\frac{mx}{1-x^2}P_l^m(x) - \frac{1}{\sqrt{1-x^2}}P_l^{m+1}(x)$$

For negative m we can make use of the identiy:

$$P_l^m(x) = (-1)^m\frac{(l+m)!}{(l-m)!}P_l^{-m}(x)$$

$$\frac{\partial}{\partial x}P_l^m(x) = (-1)^m\frac{(l+m)!}{(l-m)!}\frac{\partial}{\partial x}P_l^{-m}(x) = (-1)^m\frac{(l+m)!}{(l-m)!}\left(\frac{mx}{1-x^2}P_l^{-m}(x) - \frac{1}{\sqrt{1-x^2}}P_l^{-m+1}(x)\right)$$

for positive m we then get:

$$\frac{\partial}{\partial \phi}\left(Y_l^m(\theta,\phi)\right) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}e^{im\theta}(-sin(\phi))\left(-\frac{mcos(\phi)}{1-cos^2(\phi)}P_l^m(cos(\phi))\right)$$

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$$-\frac{1}{\sqrt{1-cos^2(\phi)}}P_l^{m+1}(cos(\phi))\Bigg)$$

$$=\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}e^{im\theta}\left(mcot(\phi)P_l^m(cos(\phi))+P_l^{m+1}(cos(\phi))\right)$$

and for negative m:

$$\frac{\partial}{\partial\phi}\left(Y_l^m(\theta,\phi)\right)=\sqrt{\frac{2l+1}{4\pi}\frac{(l+m)!}{(l-m)!}}e^{im\theta}(-sin(\phi))\left(\frac{mcos(\phi)}{1-cos^2(\phi)}P_l^{-m}(cos(\phi))\right.$$

$$\left.-\frac{1}{\sqrt{1-cos^2(\phi)}}P_l^{-m+1}(cos(\phi))\right)$$

$$=\sqrt{\frac{2l+1}{4\pi}\frac{(l+m)!}{(l-m)!}}e^{im\theta}\left(-mcot(\phi)P_l^{-m}(cos(\phi))+P_l^{-m+1}(cos(\phi))\right)$$