$$\frac{\partial \Psi^0_{lm}}{\partial \phi} = \frac{\partial}{\partial \phi}\left(\sqrt{\left(\frac{2}{na}\right)^3 \frac{(n-l-1)!}{2n[(n+l)!]^3}} \exp\left(-\frac{r}{na}\right)\left(\frac{2r}{na}\right)^l L^{2l+1}_{n-l-1}\left(\frac{2r}{na}\right) Y^m_l(\theta,\phi)\right)$$

$$= \sqrt{\left(\frac{2}{na}\right)^3 \frac{(n-l-1)!}{2n[(n+l)!]^3}} \exp\left(-\frac{r}{na}\right)\left(\frac{2r}{na}\right)^l L^{2l+1}_{n-l-1}\left(\frac{2r}{na}\right) \frac{\partial}{\partial \phi}\left(Y^m_l(\theta,\phi)\right)$$

$$\frac{\partial}{\partial \phi}\left(Y^m_l(\theta,\phi)\right) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}} e^{im\theta} \frac{\partial}{\partial \phi} P^m_l(cos(\phi))$$

$$= \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}} e^{im\theta}(-sin(\phi)) \frac{\partial}{\partial cos(\phi)} P^m_l(cos(\phi))$$

For positive m:

$$\frac{\partial}{\partial x} P^m_l(x) = \frac{\partial}{\partial x}\left((-1)^m (1-x^2)^{m/2} \frac{d^m}{dx^m} \frac{1}{2^l l!} \frac{d^l}{dx^l}\left(x^2-1\right)^l\right)$$

$$= \frac{m}{2}(-2x)(-1)^m (1-x^2)^{\frac{m}{2}-1} \frac{d^m}{dx^m} \frac{1}{2^l l!} \frac{d^l}{dx^l}\left(x^2-1\right)^l + (-1)^m (1-x^2)^{m/2} \frac{d^{m+1}}{dx^{m+1}} \frac{1}{2^l l!} \frac{d^l}{dx^l}\left(x^2-1\right)^l$$

$$= \frac{-mx}{1-x^2}(-1)^m (1-x^2)^{\frac{m}{2}} \frac{d^m}{dx^m} \frac{1}{2^l l!} \frac{d^l}{dx^l}\left(x^2-1\right)^l - \frac{1}{\sqrt{1-x^2}}(-1)^{m+1} (1-x^2)^{\frac{m+1}{2}} \frac{d^{m+1}}{dx^{m+1}} \frac{1}{2^l l!} \frac{d^l}{dx^l}\left(x^2-1\right)^l$$

$$= -\frac{mx}{1-x^2} P^m_l(x) - \frac{1}{\sqrt{1-x^2}} P^{m+1}_l(x)$$

for positive m we then get:

$$\frac{\partial}{\partial \phi}\left(Y^m_l(\theta,\phi)\right) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}} e^{im\theta}(-sin(\phi))\Bigg(-\frac{mcos(\phi)}{1-cos^2(\phi)} P^m_l(cos(\phi))$$

$$-\frac{1}{\sqrt{1-cos^2(\phi)}} P^{m+1}_l(cos(\phi))\Bigg)$$

$$= \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}} e^{im\theta}\left(mcot(\phi) P^m_l(cos(\phi)) + P^{m+1}_l(cos(\phi))\right)$$

$$\frac{\partial}{\partial \phi}\left(Y^m_l(\theta,\phi)\right) = mcot(\phi) Y^m_l(\theta,\phi) + \sqrt{(l-m)(l+m+1)} e^{-i\theta} Y^{m+1}_l(\theta,\phi)$$

For negative m we can make use of the identiy:

$$P_l^{-m}(x) = (-1)^m \frac{(l-m)!}{(l+m)!} P_l^m(x)$$

$$\frac{\partial}{\partial x} P_l^{-m}(x) = (-1)^m \frac{(l-m)!}{(l+m)!} \frac{\partial}{\partial x} P_l^m(x) = (-1)^m \frac{(l-m)!}{(l+m)!} \left(-\frac{mx}{1-x^2} P_l^m(x) - \frac{1}{\sqrt{1-x^2}} P_l^{m+1}(x) \right)$$

Using the identiy again but this time rearranged:

$$P_l^m(x) = (-1)^m \frac{(l+m)!}{(l-m)!} P_l^{-m}(x)$$

and

$$P_l^{m+1}(x) = (-1)^{m+1} \frac{(l+m+1)!}{(l-m-1)!} P_l^{-m-1}(x)$$

Therefore:

$$\frac{\partial}{\partial x} P_l^{-m}(x) = (-1)^m \frac{(l-m)!}{(l+m)!} \left(-\frac{mx}{1-x^2} (-1)^m \frac{(l+m)!}{(l-m)!} P_l^{-m}(x) - \frac{1}{\sqrt{1-x^2}} (-1)^{m+1} \frac{(l+m+1)!}{(l-m-1)!} P_l^{-m-1}(x) \right)$$

$$= -\frac{mx}{1-x^2} P_l^{-m}(x) + \frac{1}{\sqrt{1-x^2}} (l+m+1)(l-m) P_l^{-m-1}(x)$$

$$\frac{\partial}{\partial \phi} \left(Y_l^{-m}(\theta, \phi) \right) = \sqrt{\frac{2l+1}{4\pi} \frac{(l+m)!}{(l-m)!}} e^{-im\theta} (-sin(\phi)) \Bigg(-\frac{mcos(\phi)}{1-cos^2(\phi)} P_l^{-m}(cos(\phi))$$

$$+ \frac{(l+m+1)(l-m)}{\sqrt{1-cos^2(\phi)}} P_l^{-m-1}(cos(\phi)) \Bigg)$$

$$= \sqrt{\frac{2l+1}{4\pi} \frac{(l+m)!}{(l-m)!}} e^{-im\theta} \left(mcot(\phi) P_l^{-m}(cos(\phi)) - (l+m+1)(l-m) P_l^{-m-1}(cos(\phi)) \right)$$

$$= mcot(\phi) \sqrt{\frac{2l+1}{4\pi} \frac{(l+m)!}{(l-m)!}} e^{-im\theta} P_l^{-m}(cos(\phi))$$

$$-e^{i\theta} \sqrt{\frac{2l+1}{4\pi} \frac{(l+m+1)!}{(l-m-1)!} (l+m+1)(l-m)} e^{-i(m+1)\theta} P_l^{-m-1}(cos(\phi)$$

$$= mcot(\phi) Y_l^{-m} - \sqrt{(l+m+1)(l-m)} e^{i\theta} Y_l^{-(m+1)}$$