Calculation of $\chi^0(q,\omega)$

This is a sketch of the derivation of the leading term in the expansion of the transverse susceptibility for the ferromagnetic Hubbard model. The constant factors involving the permeability, the Lande-g factor etc. are all omitted (or set equal to unity).

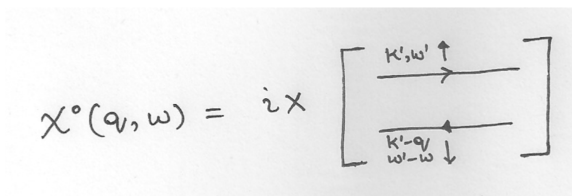
The (Fourier transformed) transverse susceptibility is given by

$$\chi^{-+}(q,\omega) = \frac{\chi^0(q,\omega)}{1 - U\chi^0(q,\omega)} \tag{1}$$

where

$$\chi^0(q,\omega) = i\sum_{k_1}\int_{-\infty}^{\infty}\frac{d\omega_1}{2\pi}G^0_\uparrow(k_1,\omega_1)G^0_\downarrow(k_1-q,\omega_1-\omega) \tag{2}$$

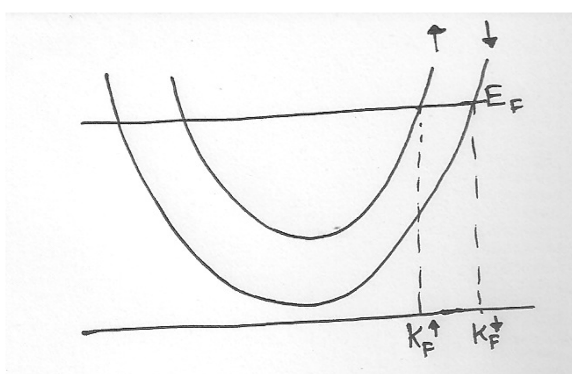
corresponds to the Feynman diagram:



At this level, no self-energy (magnon) corrections are being considered. The form of the Fourier transformed time ordered Green's function in general is

$$G^0_\sigma(k,\omega) = \frac{\theta(k-k_F^\sigma)}{\omega - E_k^\sigma + i\eta} + \frac{\theta(k_F^\sigma - k)}{\omega - E_k^\sigma - i\eta} \tag{3}$$

where $\sigma = \uparrow / \downarrow$. Note that k_F^σ is defined as the solution to the equation $E^\sigma(k) = E_F$, as shown in the diagram.

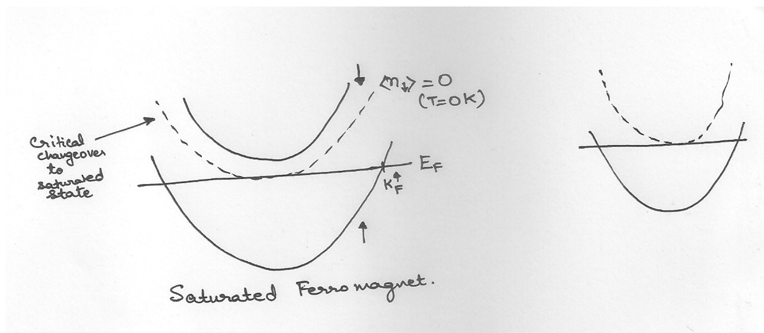


Substituting the forms of the Green's function for up-spin and down-spin into the expression for $\chi^0(q, \omega)$ and observing that the contributions will come only from terms which have poles on different sides of the real ω' axis, we get the expression

$$\chi^0(q,\omega) = -\sum_{k_1}\left[\frac{\theta(k_1 - k_F^{\uparrow})\theta(k_F^{\downarrow} - |k_1 - q|)}{\omega + E^{\downarrow}_{k_1-q} - E^{\uparrow}_{k_1} + i\eta} + \frac{\theta(|k_1 - q| - k_F^{\downarrow})\theta(k_F^{\uparrow} - k_1)}{E^{\uparrow}_{k_1} - \omega - E^{\downarrow}_{k_1-q} - i\eta}\right] \tag{4}$$

$$= \sum_{k_1}\left[\frac{\theta(k_1 - k_F^{\uparrow})\theta(k_F^{\downarrow} - |k_1 - q|)}{E^{\uparrow}_{k_1} - E^{\downarrow}_{k_1-q} - \omega - i\eta} + \frac{\theta(|k_1 - q| - k_F^{\downarrow})\theta(k_F^{\uparrow} - k_1)}{E^{\downarrow}_{k_1-q} - E^{\uparrow}_{k_1} + \omega - i\eta}\right] \tag{5}$$

But in case of a **saturated** ferromagnet where the down-spin band is completely above the Fermi level E_F, the variable $k_F^{\downarrow}$ is complex as the down-spin band does not intersect the Fermi level. The situation looks like this:



The way to reconcile this seems to be to consider the limiting case of the down-spin band moving up. The transition from real $k_F^{\downarrow}$ to complex $k_F^{\downarrow}$ occurs through $k_F = 0$ (I am assuming the minima of the bands are at $k = 0$, as the bands are parabolic and aligned, and we can shift the k axis suitably.)

In other words, $k_F^{\downarrow} = 0$ is the "last real value" that the system as it becomes a completely saturated ferromagnet. In order to use the Green's function in the form given in (3), it we can set $k_F^{\downarrow} = 0$ and obtain the following expression for the down-spin Green's function

$$G^0(k_1 - q, \omega_1 - \omega) = \frac{\theta(|k_1 - q|)}{\omega_1 - \omega - E^{\downarrow}_{k_1-k} + i\eta} \tag{6}$$

So, for a saturated ferromagnet,

$$\chi^0_{sat}(q,\omega) = i\sum_{k_1}\int_{-\infty}^{\infty}\frac{d\omega_1}{2\pi}G^0_{\uparrow}(k_1,\omega_1)G^0_{\downarrow}(k_1 - k, \omega_1 - \omega)$$

$$= i\sum_{k_1}\int_{-\infty}^{\infty}\frac{d\omega_1}{2\pi}\left[\frac{\theta(k_1 - k_F^{\uparrow})}{\omega_1 - E^{\uparrow}_{k_1} + i\eta} + \frac{\theta(k_F^{\uparrow} - k_1)}{\omega_1 - E^{\uparrow}_{k_1} - i\eta}\right]\left[\frac{\theta(|k_1 - q|)}{\omega_1 - \omega - E^{\downarrow}_{k_1-q} + i\eta}\right] \tag{7}$$

Again, by the reasoning used for the unsaturated ferromagnet, in such an expression, only the terms containing poles on different sides of the ω_1 axis will contribute. Hence, having picked a $+i\eta$ term for the down-spin Green's function, only the second term in the up-spin Green's function contributes a non-zero value, and we end up with

$$\chi^0_{sat}(q,\omega) = i\sum_{k_1}\int_{-\infty}^{\infty}\frac{d\omega_1}{2\pi}\frac{\theta(k_F^{\uparrow}-k_1)\theta(|k_1-q|)}{(\omega_1-E^{\uparrow}_{k_1}-i\eta)(\omega_1-\omega-E^{\downarrow}_{k_1-q}+i\eta)} \tag{8}$$

$$= -\sum_{k_1}\frac{\theta(k_F^{\uparrow}-k_1)}{E^{\uparrow}_{k_1}-\omega-E^{\downarrow}_{k_1-q}+i\eta} \tag{9}$$

$$= \sum_{k_1}\frac{\theta(k_F^{\uparrow}-k_1)}{\omega+(E^{\downarrow}_{k_1-q}-E^{\uparrow}_{k_1})-i\eta} \tag{10}$$

Note that this expression could have been obtained directly from the expression for $\chi^0(q,\omega)$ for a general ferromagnet (with $k_F^{\downarrow}=0$). Using $E^{\downarrow}_k = E_k+\Delta$ and $E^{\uparrow}_k = E_k-\Delta$, we get

$$\boxed{\chi^0_{sat}(q,\omega) = \sum_{k_1}\frac{\theta(k_F^{\uparrow}-k_1)}{\omega+2\Delta+(E_{k_1-q}-E_{k_1})-i\eta}} \tag{11}$$

The sum should be restricted to up-spin states with energy less than the Fermi energy E_F, which is already incorporated into this expression due to the θ-function.