We intend to prove that the recurrence sequence

$$\begin{aligned} n_1 &= 1 \\ n_2 &= 17 \\ n_3 &= 241 \\ n_i &= 15n_{i-1} - 15n_{i-2} + n_{i-3} \ , \qquad \text{for } i > 3 \end{aligned}$$

with values 1, 17, 241, 3361, 46817, ..., produces the *only* positive values of n such that $(3n-1)(n+1)$ is a square.

First, we will express this sequence in a different form. Given

$$\begin{aligned} a_0 &= -1 \\ a_1 &= 1 \\ a_i &= 4a_{i-1} - a_{i-2} \ , \qquad \text{for } i > 1 \end{aligned}$$

with values -1, 1, 5, 19, 71, 265, ..., and

$$\begin{aligned} b_0 &= 1 \\ b_1 &= 1 \\ b_i &= 4b_{i-1} - b_{i-2} \ , \qquad \text{for } i > 1 \end{aligned}$$

with values 1, 1, 3, 11, 41, 153, ..., we see that $n_i = \dfrac{a_i^2 + b_i^2}{2}$ for $i > 0$: indeed,

$$\begin{aligned} n_1 &= \frac{a_1^2 + b_1^2}{2} = \frac{1^2 + 1^2}{2} = 1 \\ n_2 &= \frac{a_2^2 + b_2^2}{2} = \frac{5^2 + 3^2}{2} = 17 \\ n_3 &= \frac{a_3^2 + b_3^2}{2} = \frac{19^2 + 11^2}{2} = 241 \end{aligned}$$

and, for $i > 3$,

$$\begin{aligned} n_i &= 15n_{i-1} - 15n_{i-2} + n_{i-3} \\ &= 15 \cdot \frac{a_{i-1}^2 + b_{i-1}^2}{2} - 15 \cdot \frac{a_{i-2}^2 + b_{i-2}^2}{2} + \frac{a_{i-3}^2 + b_{i-3}^2}{2} \\ &= \frac{15(4a_{i-2} - a_{i-3})^2 + 15(4b_{i-2} - b_{i-3})^2 - 15a_{i-2}^2 - 15b_{i-2}^2 + a_{i-3}^2 + b_{i-3}^2}{2} \\ &= \frac{225a_{i-2}^2 - 120a_{i-2}a_{i-3} + 16a_{i-3}^2 + 225b_{i-2}^2 - 120b_{i-2}b_{i-3} + 16b_{i-3}^2}{2} \\ &= \frac{(15a_{i-2} + 4a_{i-3})^2 + (15b_{i-2} + 4b_{i-3})^2}{2} \\ &= \frac{(4(4a_{i-2} - a_{i-3}) - a_{i-2})^2 + (4(4b_{i-2} - b_{i-3}) - b_{i-2})^2}{2} \\ &= \frac{(4a_{i-1} - a_{i-2})^2 + (4b_{i-1} - b_{i-2})^2}{2} \\ &= \frac{a_i^2 + b_i^2}{2} \end{aligned}$$

Now we need the following

LEMMA 1:

If r, s, n are integers such that

$$\begin{cases} r^2 + s^2 = 2n \\ r^2 - s^2 = n - 1 \end{cases}$$

then $(3n-1)(n+1)$ is a square.

Proof: Adding the two equations in the above system produces $2r^2 = 3n - 1$, and subtracting them yields $2s^2 = n + 1$. The product of both is

$$4r^2s^2 = (3n-1)(n+1)$$

which is a square (namely, the square of $2rs$). ■

LEMMA 2:

The values a_i and b_i, from the sequences defined earlier, satisfy

$$\begin{aligned} &3b_i^2 - a_i^2 = 2 && \text{for all } i \geq 0 \\ &3b_i b_{i-1} - a_i a_{i-1} = 4 && \text{for all } i > 0 \end{aligned}$$

Proof: By induction on i. With $i = 0$, $3b_0^2 - a_0^2 = 3 - 1 = 2$; and with $i = 1$, $3b_1^2 - a_1^2 = 3 - 1 = 2$ and $3b_1b_0 - a_1a_0 = 3 + 1 = 4$.

For the induction step we assume, for $k \geq 2$,

$$\begin{aligned} 3b_{k-1}^2 - a_{k-1}^2 &= 2 \\ 3b_{k-2}^2 - a_{k-2}^2 &= 2 \\ 3b_{k-1}b_{k-2} - a_{k-1}a_{k-2} &= 4 \end{aligned}$$

and try to prove the statements for $i = k$.

Indeed,

$$\begin{aligned} 3b_k^2 - a_k^2 &= 3(4b_{k-1} - b_{k-2})^2 - (4a_{k-1} - a_{k-2})^2 \\ &= 48b_{k-1}^2 - 24b_{k-1}b_{k-2} + 3b_{k-2}^2 - 16a_{k-1}^2 + 8a_{k-1}a_{k-2} - a_{k-2}^2 \\ &= 16(3b_{k-1}^2 - a_{k-1}^2) - 8(3b_{k-1}b_{k-2} - a_{k-1}a_{k-2}) + (3b_{k-2}^2 - a_{k-2}^2) \\ &= 16 \cdot 2 - 8 \cdot 4 + 2 = 2 \end{aligned}$$

Also,

$$\begin{aligned} 3b_kb_{k-1} - a_ka_{k-1} &= 3(4b_{k-1} - b_{k-2})b_{k-1} - (4a_{k-1} - a_{k-2})a_{k-1} \\ &= 12b_{k-1}^2 - 3b_{k-1}b_{k-2} - 4a_{k-1}^2 + a_{k-1}a_{k-2} \\ &= 4(3b_{k-1}^2 - a_{k-1}^2) - (3b_{k-1}b_{k-2} - a_{k-1}a_{k-2}) \\ &= 4 \cdot 2 - 4 = 4\,. \end{aligned}$$ ■

Next, we prove that the values of n_i make $(3n_i - 1)(n_i + 1)$ a square.

We will proceed by proving a different statement: that, for all $i > 0$, the system

$$\begin{cases} a_i^2 + b_i^2 = 2c \\ a_i^2 - b_i^2 = c - 1 \end{cases}$$

is satisfied by setting $c = n_i$. As a consequence of Lemma 1, $(3n_i - 1)(n_i + 1)$ will be a square for all $i > 0$.

From the identity $n_i = \dfrac{a_i^2 + b_i^2}{2}$ we know that $c = n_i$ satisfies the first equation,

$$a_i^2 + b_i^2 = 2n_i$$

From Lemma 2 we know that, for all $i \geq 0$,

$$3b_i^2 - a_i^2 = 2$$

Subtracting these two we obtain

$$\begin{aligned} 2a_i^2 - 2b_i^2 &= 2n_i - 2 \\ a_i^2 - b_i^2 &= n_i - 1 \end{aligned}$$

and $c = n_i$ satisfies also the second equation. $\blacksquare$

Now, it remains to be proven that, if $(3n-1)(n+1)$ is a square for some positive integer n, then n is necessarily one of the values from the sequence n_i.

A sketch in this direction (with two big holes) can be:

- n is an odd number; for, if $n = 2k$ for some integer k, then

  $$(6k-1)(2k+1) = 12k^2 + 4k - 1$$

  would need to be a square; but this quantity is congruent to 3 modulo 4, while all squares are congruent modulo 4 to either 0 or 1.

- If n is odd, both $(3k-1)$ and $(n+1)$ are even; therefore we can write, for some positive integers r, s,

  $$\begin{aligned} 3n - 1 &= 2r \\ n + 1 &= 2s \end{aligned}$$

  It would help to prove that r and s are coprime squares, and then have

  $$\begin{aligned} 3n - 1 &= 2a^2 \\ n + 1 &= 2b^2 \end{aligned}$$

- At this point we could arrive at the Pell-like equation

  $$a^2 - 3b^2 = -2$$

  and it would help to know that the only positive solutions from this equation come necessarily from the sequences a_i and b_i. The key to this may be the fact that these two sequences appear in the continued fraction for $\sqrt{3}$:

  $$\sqrt{3} = [1; 1, 2, 1, 2, ...]$$

  where it is not hard to prove that the convergents $\dfrac{p_k}{q_k}$ satisfy, for k odd ≥ 3,

  $$\begin{aligned} p_{2n+1} &= a_n \qquad\qquad \text{for } n \geq 1 \\ q_{2n+1} &= b_n \end{aligned}$$