Green's Function for Regular Sturm-Liouville Problems

We are interested in in solving problems like

$$Ly := (py')' - qy = f \tag{1}$$

$$B_1 y = \beta_1 y(a) + \gamma_1 y'(a) \tag{2}$$

$$B_2 y = \beta_2 y(b) + \gamma_2 y'(b). \tag{3}$$

To this end we define the operator

$$Ly = (py')' - qy$$

under the assumption that $\lambda = 0$ is not an eigenvalue of L and where p, p', and q are continuous on $[a, b]$, $p(x) > 0$ on $[a, b]$ and $|\gamma_j| + |\beta_j| \neq 0$ for $j = 1, 2$, in the Hilbert space $H = L^2(a, b)$ with inner product

$$\langle f, g\rangle = \int_a^b f(x)\overline{g(x)}\, dx$$

and the induced norm

$$\|f\|^2 = \int_a^b |f(x)|^2\, dx.$$

To do this we first need a couple of basic results from the theory of ordinary differential equations. One is a consequence of the fundamental existence uniqueness theorem for ordinary differential equations which guarantees the unique existence of a solution to the following initial value problem

$$Ly = (py')' - qy = 0 \tag{4}$$

$$y(a) = \alpha_1$$

$$y'(a) = \alpha_2$$

for any α_1 and α_2. Among other things this allows us to ensure that we can find two linearly independent solutions to the equation (1) by taking functions u_1 and u_2 satisfying

$$Lu_j = (pu_j')' - qu_j = 0 \tag{5}$$

$$u_j(a) = \begin{cases} 1 & j = 1 \\ 0 & j = 2 \end{cases}$$

$$u_j'(a) = \begin{cases} 0 & j = 1 \\ 1 & j = 2 \end{cases}$$

Now it is useful to rewrite equation (1) as follows. Expand the first term and divide by p to get

$$y'' + \frac{p'}{p}y' - \frac{q}{p}y = 0$$

Now let us denote $P = p'/p$ and $Q = q/p$ to get

$$y'' + Py' - Qy = 0. \tag{6}$$

For this equation we derive the well known Abel formula for the Wronskian. Let y_j $j = 1, 2$ be two linearly independent solutions of (5). Then the wronskian is defined by

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}.$$

Then Abel's formula is

$$W(x) = W(a) \exp\left(-\int_a^x P(\xi)\, dx\right) \tag{7}$$

To verify this we expand the Wronskian determinant and differentiate to obtain

$$\begin{aligned} W'(x) &= (y_1 y_2' - y_1' y_2)' \qquad (8) \\ &= y_1 y_2'' + y_1' y_2' - y_1' y_2' - y_1 y_2'' \\ &= y_1 y_2'' - y_1 y_2'' \\ &= y_1(-P y_2' + Q y_2) - y_1(-P y_1' + Q y_1) \\ &= -P(y_1 y_2' - y_1' y_2) = -PW(x). \end{aligned}$$

Thus we consider the differential equation

$$W' = -PW$$

which is readily solved by separation of variables. At this point we also recall that $P = p'/p$ We have

$$\frac{dW}{W} = -\frac{dp}{p}$$

so that

$$\ln|W| = -\ln|p| + C$$

which gives $C = W(a)p(a)$ and we have

$$W(x) = W(a)\frac{p(a)}{p(x)}. \tag{9}$$

Taking into account $P = p'/p$ this is exactly what we get in (6).

Lemma 1. *Under our assumption that $\lambda = 0$ is not an eigenvalue of L, it is always possible to find a basis of solutions u_j of $Lu = 0$ satisfying $B_j u_j = 0$ and $B_i u_j = 0$ for $i = 1, 2$ and $i \neq j$.*

To see this we note that this assumption implies that we cannot find a function $u \neq 0$ such that $B_1 u = 0$ and $B_2 u = 0$. So we first choose the functions y_j in (5) and note that every solution of $Ly = 0$ can then be written in the form $u = c_1 y_1 + c_2 y_2$. Now we find c_1, c_2 so that $B_1 u = 0$. We have

$$0 = B_1 u = \gamma_1 (c_1 y_1'(a) + c_2 y_2'(a)) + \beta_1 (c_1 y_1(a) + c_2 y_2(a)) = \gamma_1 c_2 + \beta_1 c_1.$$

This is easily satisfied by $\gamma_1 = -\beta_1$ and $c_2 = \gamma_2$ so we obtain

$$u_1(x) = -\beta_1 y_1(x) + \gamma_2 y_2(x).$$

Note that by our assumption $B_2(u_1) \neq 0$.

For u_2 we need to do a bit more work. Once again we look for u in the form $u = c_1 y_1 + c_2 y_2$.

$$\begin{aligned} 0 = B_1 u &= \gamma_2 (c_1 y_1'(b) + c_2 y_2'(b)) + \beta_2 (c_1 y_1(b) + c_2 y_2(b)) \\ &= c_1 \left[\gamma_2 y_1'(b) + \beta_2 y_1(b)\right] + c_2 \left[\gamma_2 y_2'(b) + \beta_2 y_2(b)\right] \\ &= c_1 (B_2 y_1) + c_2 (B_2 y_2) \end{aligned}$$

Then, for example, we can set $c_1 = -B_2 y_2$ and $c_2 = B_2 y_1$ and we have

$$u_2(x) = -(B_2 y_2) y_1(x) + (B_2 y_1) y_2(x).$$

A Green's function for the problem (1)-(3) is a function satisfying

Definition 1. A Green's function is a function $g(x, \xi)$ for $(x, \xi) \in [a, b] \times [a, b]$ such that

1. The following hold
   (a) $g(\cdot, \cdot)$ is continuous on $[a, b] \times [a, b]$,
   (b) $\dfrac{\partial g}{\partial x}(\cdot, \xi)$ is continuous on $[a, \xi) \times (\xi, b]$, and,
   (c) $\left.\dfrac{\partial g(x, \xi)}{\partial x}\right|_{x=\xi^-}^{x=\xi^+} \equiv \dfrac{\partial g}{\partial x}(\xi^+, \xi) - \dfrac{\partial g}{\partial x}(\xi^-, \xi) = \dfrac{1}{p(\xi)}$
2. for all $\xi \in [a, b]$, $g(x, \xi)$ solves $L(g) = 0$, $x \neq \xi$.
3. for all $\xi \in (a, b)$, $B_i(g) = 0$.

Let us first construct the Green's function and then we will show that it does indeed lead to a formula for the inverse of L. One way to construct the Green's function is to use the properties given in Definition 1. To do this we first construct the functions u_j in Lemma 1.

We then seek g in the following form

$$g(x,\xi) = \begin{cases} Au_1(x), & a \le x \le \xi \\ \\ Bu_2(x), & \xi < x < b \end{cases}.$$

In order to satisfy the Assumption 1 of Definition 1 (continuity) we need

$$Au_1(\xi) - Bu_2(\xi) = 0.$$

To satisfy the jump condition we would need

$$Bu_2'(\xi) - Au_1'(\xi) = \frac{1}{p(\xi}.$$

This gives a system of two equations in two unknowns with determinant of the coefficient coefficient given by

$$\Delta = \begin{vmatrix} u_1(\xi) & -u_2(\xi) \\ u_1'(\xi) & -u_2'(\xi) \end{vmatrix} = -W$$

(the Wronskian). Using Cramer's rule we can find A and B as

$$A = -\frac{1}{W} \begin{vmatrix} 0 & -u_2(\xi) \\ -1/p(\xi) & -u_2'(\xi) \end{vmatrix} = \frac{u_2(\xi)}{p(\xi)W(\xi)}$$

and

$$B = -\frac{1}{W} \begin{vmatrix} u_1(\xi) & 0 \\ u_1'(\xi) & -1/p(\xi) \end{vmatrix} = \frac{u_1(\xi)}{p(\xi)W(\xi)}$$

We have computed g in the form

$$g(x,\xi) = \frac{1}{p(\xi)W(\xi)} \begin{cases} u_1(x)u_2(\xi), & a \le x \le \xi \\ \\ u_1(\xi)u_2(x), & \xi < x < b \end{cases}.$$

Next we recall that using Abel's formula (see (9)) we can simplify this formula since $p(\xi)W(\xi) = p(a)W(a)$ for all ξ, so we arrive at the final formula.

$$g(x,\xi) = \frac{1}{p(a)W(a)} \begin{cases} u_1(x)u_2(\xi), & a \le x \le \xi \\ \\ u_1(\xi)u_2(x), & \xi < x < b \end{cases}. \tag{10}$$

Theorem 1. *The operator K on H defined by*

$$K\varphi = \int_a^b g(x,\xi)\varphi(\xi)\, d\xi$$

is a compact operator. Furthermore, it is self-adjoint since

$$g(x,\xi) = g(\xi, x).$$

The proof of this result follows from a pair of Lemmas.

Lemma 2. *The collection of functions*

$$\varphi_k(x) = (b-a)^{-1/2} \exp\left(2\pi i k \frac{(x-a)}{(b-a)}\right), \quad k = 0, \pm 1, \pm 2, \cdots$$

forms an orthonormal basis in $L^2(a,b)$. *Furthermore, the functions*

$$\psi_{k,j}(x,\xi) = \varphi_k(x)\overline{\varphi_j(\xi)} \quad k,j = -\infty, \cdots, \infty$$

form an othonormal basis for $L^2([a,b]\times[a,b])$. *Indeed, if* $g(x,\xi) \in L^2([a,b]\times[a,b])$ *then*

$$g(x,\xi) = \sum_{k,j=-\infty}^{\infty} c_{k,j}\psi_{k,j}(x,\xi)$$

where

$$\sum_{k,j=-\infty}^{\infty} |c_{k,j}|^2 < \infty$$

and

$$c_{k,j} = \iint_{-\infty}^{\infty} \psi_{k,j}(x,\xi) g(x,\xi)\, d\xi dx.$$

Proof: After a change of variables mapping the interval $[a,b]$ to the interval $[0,2\pi]$ the functions are precisely the basis that gives the Fourier series for functions defined on $[0,2\pi]$. We need only note that if $f \perp \varphi_k$ for all k implies that $f = 0$ since

$$\int_a^b f(x)\varphi_k(x)\, dx = (b-a)^{-1/2} \int_a^b f(x) \exp\left(2\pi i k \frac{x-a}{b-a}\right) dx = \int_0^1 e^{2\pi i k s} f(a+(b-a)s)\, ds$$

where we have changed variables using $s = (x-a)/(b-a)$. Let us denote $g(s)$ defined on $[0,1]$ by $g(s) = f(a+(b-a)s)$, then $f \perp \varphi_k$ is equivalent to

$$\int_0^1 e^{2\pi i k s} g(s)\, ds = 0 \quad k = 0, \pm 1, \pm 2, \cdots.$$

By elementary Fourier series we know that $g(s) = 0$ a.e. and therefore $f = 0$ in $L^2(a,b)$. The second part follows from a well know result on multivariable Fourier series and the fact that

$$L^2([a,b]\times[a,b]) = L^2[a,b] \otimes L^2[a,b].$$

$\square$

Lemma 3. *If $k(x,t)$ satisfies*

$$\int_a^b \int_a^b |k(x,\xi)|^2 \, d\xi dx < \infty$$

then the operator K

$$K\varphi = \int_a^b k(x,\xi)\varphi(\xi)\, d\xi$$

is a compact operator in $L^2(a,b)$.

Proof: Using Lemma 2 we have

$$k(x,\xi) = \sum_{k,j=-\infty}^{\infty} \langle k, \psi_{k,j}\rangle \psi_{k,j}(x,\xi).$$

Furthermore, we have

$$k_N(x,\xi) = \sum_{|k|,|j|\le N} \langle k, \psi_{k,j}\rangle \psi_{k,j}(x,\xi)$$

satisfies

$$\|k_N - k\|_{L^2(a,b)} \to 0 \quad N \to \infty.$$

Since

$$\sum_{k,j=-\infty}^{\infty} |\langle k, \psi_{k,j}\rangle|^2 < \infty$$

for a given $\epsilon > 0$ we can find N so that

$$|\langle k, \psi_{k,j}\rangle| < \epsilon \quad \text{for all} \quad |k|, |j| > N+1.$$

Therefore, the finite rank operators

$$K_N\varphi(x) = \int_a^b k_N(x,\xi)\varphi(\xi)\, d\xi = \sum_{|k|,|j|\le N} \langle k, \psi_{k,j}\rangle \langle \varphi, \varphi_j\rangle \varphi_k(x)$$

satisfy

$$\begin{aligned}
\|(K_N - K)(\varphi)\|^2 &= \left\| \sum_{|k,j|\ge N+1} \langle k, \psi_{k,j}\rangle \langle \varphi, \varphi_j\rangle \varphi_k(x) \right\|^2 \\
&= \left\langle \sum_{|k,j|\ge N+1} \langle k, \psi_{k,j}\rangle \langle \varphi, \varphi_j\rangle \varphi_k(x), \sum_{|k,j|\ge N+1} \langle k, \psi_{k,j}\rangle \langle \varphi, \varphi_j\rangle \varphi_k(x) \right\rangle \\
&= \sum_{|k|,|j|\ge N+1} |\langle k, \psi_{k,j}\rangle|^2 |\langle \varphi, \varphi_k\rangle|^2 \\
&< \epsilon \|\varphi\|^2
\end{aligned}$$

and we have

$$\|K_N - K\| \xrightarrow{N\to\infty} 0,$$

and as a uniform limit of finite rank operators we see that K is compact. $\square$

An important feature of a Green's function lies in the following. In order to solve

$$(L-\lambda)u = f, \quad B_1(u) = 0, \quad B_2(u) = 0$$

we want to find

$$u = (L-\lambda)^{-1} f.$$

But then we see that finding $(L-\lambda)^{-1}$ amounts to solving $(L-\lambda)u = f, \quad B_1(u) = 0, \quad B_2(u) = 0$ which is a boundary value problem. Notice that the presence of λ is, on the one hand artificial, and on the other hand quite important. Notice that we could eliminate the term involving λ by simply redefining $\widetilde{q} = q + \lambda$ and the defining $L_\lambda = (py')' - \widetilde{q}y$. The problem to solve then becomes

$$L_\lambda u = f, \quad B_1(u) = 0, \quad B_2(u) = 0.$$

So what is the difference. Well remember the "small assumption" that zero is not an eigenvalue of L. For general λ zero could be an eigenvalue of L_λ. Indeed, the set of all eigenvalues λ for L are extremely important numbers in practical problems. Under the assumption that $\lambda = 0$ is not an eigenvalue of L what we will show is that L^{-1} is compact and therefore has at most countable collection of eigenvalues μ_j whose only accumulation point is zero. Then we see, from the spectral mapping theorem, that the eigenvalues of L are precisely $\lambda_j = 1/\mu_j$ which tend to infinity.

We now turn to the main application of Green's function in this section. Namely, we consider the nonhomogeneous BVP.

$$L_\lambda(y) = (py')' - q(x)y + \lambda y = f(x), \qquad a < x < b$$

$$B_1(y) = 0, \quad B_2(y) = 0$$

where

$$B_1 y = \beta_1 y(a) + \gamma_1 y'(a), \quad B_2 y = \beta_2 y(b) + \gamma_2 y'(b),$$

and $k \in C^1(a,b), p(x) > 0, x \in [a,b]$.

First we recall a classical formula whose general counterpart has far reaching consequences in the theory of ordinary and partial differential equations and the theory of weak solutions. At this point we will only consider a very special case. Namely, given any two functions u and v, a straightforward calculation gives the so-called Lagrange Identity:

$$vL_\lambda(u) - uL_\lambda(v) = \frac{d}{dx}P(u,v)$$

where

$$P(u,v) = p\,(u'v - uv')$$

and we note that integration gives the so-called Green's formula

$$\int_a^b [vL_\lambda(u) - uL_\lambda(v)] = P(u,v)|_{x=a}^{x=b}$$

Let $g(x,\xi)$ denote the green's function for the homogeneous problem (1)-(3). From Lagrange's identity, for $x \neq \xi$

$$g(x,\xi)L_\lambda(y) - yL_\lambda(g(x,\xi)) = \frac{d}{dx}[p(gy' - yg')]$$

which implies

$$\int_a^{\xi^-} gL_\lambda(y)dx = p(gy' - g'y)]_a^{\xi^-}$$

and

$$\int_{\xi^+}^b gL_\lambda(y)dx = p(gy' - g'y)]_{\xi^+}^b.$$

Hence

$$\int_a^b gL_\lambda(y)dx = p(gy' - g'y)]_a^b - p(gy' - g'y)]_{\xi^-}^{\xi^+}.$$

Note that our boundary conditions B_1, B_2 have the property that if u, v satisfy $B_1(u) = 0 = B_2(v)$, then

$$[p(gy' - g'y)]_a^b = 0$$

Thus we have

$$\begin{aligned}
\int_a^b gL_\lambda(y)dx &= -\left[p(gy' - g'y)\right]_{\xi^-}^{\xi^+} \\
&= p\left[\frac{\partial g}{\partial x}(\xi^+,\xi) - \frac{\partial g}{\partial x}(\xi^-,\xi)\right] y(\xi) \\
&= y(\xi).
\end{aligned}$$

Therefore if y satisfies $L(y) = f$, then we should have $y(x) = \displaystyle\int_a^b g(x,\xi)f(\xi)\,d\xi$. Thus we have

$$\int_a^b g(x,\xi)L_\lambda(y)(x)dx = \int_a^b L_\lambda g(x,\xi)(y)(x)dx = y(\xi)$$

which suggest that $L_\lambda g(x,\xi) = \delta(x-\xi)$, i.e., the solution to

$$L_\lambda(y) = f$$
$$B_i(y) = 0$$

would be given by

$$y(x) = \int_a^b g(x,\xi)f(\xi)d\xi$$

provided that λ is not an eigenvalue.

References

[1] L. Bers, F. John, M. Schechter, *Partial Differential Equations*, Interscience, 1964.

[2] Neilson Dunford, Jacob T. Schwartz *Linear Operators, Part 1, General Theory*, Wiley.

[3] R. Kress, *Linear Integral Equations*, 2nd Edition, Springer, New York, 1999.

[4] P. Garrett, Functions on the Circle, (Online Notes) at `www.math.umn.edu/~garrett/m/mfms/notes/09_sobolev.ps`

[5] Naylor, Arch W., Sell, George R., *Linear Operator Theory in Engineering and Science*, Series: Applied Mathematical Sciences , Vol. 40 1st ed. 1982. 2nd printing, 2000.

[6] Frigyes Riesz, Béla Sz.-Nagy, *Functional Analysis*, Dover.

[7] R.E. Edwards, *Functional Analysis: Theory and Applications* , Dover.

[8] Georgi E. Shilov, *Elementary Functional Analysis*, Dover.

[9] George Bachman, Lawrence Narici *Functional Analysis*

[10] Kosaku Yoshida, *Functional analysis*, Springer.