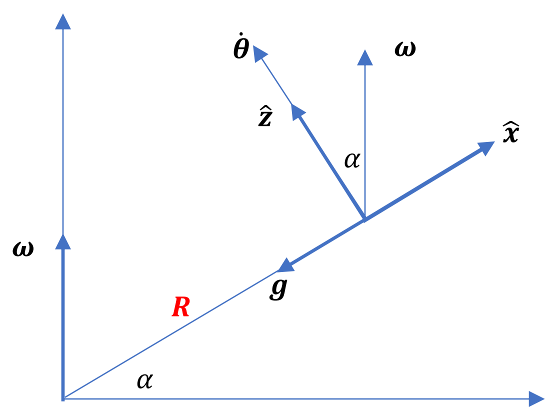
7-11 A gyroscope consists of a wheel of radius r, all of whose mass is located on the rim. The gyroscope is rotating with angular velocity $\dot{\theta}$ about its axis, which is horizontal and is fixed relative to the earth's surface. We choose a coordinate system at rest relative to the earth whose z axis coincides with the gyroscope axis and whose origin lies at the center of the wheel. The angular velocity ω of the earth lies in the xz plane, making an angle α with the gyroscope axis.

Find the x, y, and z components of the torque $\boldsymbol{N}$ about the origin, due to the Coriolis force in the xyz coordinate system, acting on a mass m on the rim of the gyroscope wheel whose polar coordinates in the xy plane are r, θ. Use this result to show that the total Coriolis torque on the gyroscope, if the wheel has a mass M, is

$$\boldsymbol{N} = -\frac{1}{2}\boldsymbol{j}Mr^2\omega\dot{\theta}sin\alpha$$

This equation is the basis for the operation of the gyrocompass.

The coordinate system in the rotating frame can be set up by defining a plane perpendicular to the radial direction in the region of interest with $\hat{\boldsymbol{z}}$ in the north direction and $\hat{\boldsymbol{x}}$ in the vertical direction, thus placing $\boldsymbol{\omega}$ in the xz plane. The east direction perpendicular to the xz plane is designated by $\hat{\boldsymbol{y}}$ thus forming a righthand coordinate system.



$$\boldsymbol{\omega} = \omega(sin\alpha\hat{\boldsymbol{x}} + cos\alpha\hat{\boldsymbol{z}})$$

$$\dot{\boldsymbol{\theta}} = \dot{\theta}\hat{\boldsymbol{z}}$$

$$\boldsymbol{R} = R\hat{\boldsymbol{x}}$$

$$\boldsymbol{g} = -g\hat{\boldsymbol{x}}$$

A mass m in the xy plane has polar coordinates r, θ. The velocity of m is

$$\boldsymbol{v}_m = \dot{\boldsymbol{\theta}} \times \boldsymbol{r}$$

$$\boldsymbol{r} = r(cos\theta\hat{\boldsymbol{x}} + sin\theta\hat{\boldsymbol{y}})$$

$$\boldsymbol{v}_m = \dot{\theta}\hat{\boldsymbol{z}} \times r(cos\theta\hat{\boldsymbol{x}} + sin\theta\hat{\boldsymbol{y}})$$

$$\boldsymbol{v}_m = r\dot{\theta}(cos\theta\hat{\boldsymbol{y}} - sin\theta\hat{\boldsymbol{x}})$$

The equation of motion in the moving frame is, from Eq. (7.43),

$$m\frac{d^{*^2}\boldsymbol{r}}{dt} = m[\boldsymbol{g} - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r})] - 2m\boldsymbol{\omega} \times \frac{d^*\boldsymbol{r}}{dt}$$

The torque due to gravity is zero because it acts through the center of mass of the gyroscope and the rim lies in the xz plane. The effective gravity is very small and can be ignored. Therefore, the only significant force acting on the mass to produce a torque is the Coriolis force. So, only this force is required.

$$\frac{d^*\boldsymbol{r}}{dt} = \boldsymbol{v}_m$$

$$m\frac{d^{*^2}\boldsymbol{r}}{dt} = -2m\boldsymbol{\omega} \times \boldsymbol{v}_m$$

$$m\frac{d^{*^2}\boldsymbol{r}}{dt} = -2m\omega(sin\alpha\hat{\boldsymbol{x}} + cos\alpha\hat{\boldsymbol{z}}) \times r\dot{\theta}(cos\theta\hat{\boldsymbol{y}} - sin\theta\hat{\boldsymbol{x}})$$

$$\boldsymbol{F}_m = m\frac{d^{*^2}\boldsymbol{r}}{dt} = -2m\omega r\dot{\theta}(cos\theta sin\alpha\hat{\boldsymbol{z}} - cos\theta cos\alpha\hat{\boldsymbol{x}} - sin\theta cos\alpha\hat{\boldsymbol{y}})$$

The torque on this mass is

$$\boldsymbol{N}_m = \boldsymbol{r} \times \boldsymbol{F}_m$$

$$\boldsymbol{N}_m = -2m\omega r^2\dot{\theta}(cos\theta\hat{\boldsymbol{x}} + sin\theta\hat{\boldsymbol{y}}) \times (cos\theta sin\alpha\hat{\boldsymbol{z}} - cos\theta cos\alpha\hat{\boldsymbol{x}} - sin\theta cos\alpha\hat{\boldsymbol{y}})$$

$$\boldsymbol{N}_m = -2m\omega r^2\dot{\theta}(-cos^2\theta sin\alpha\hat{\boldsymbol{y}} - cos\theta sin\theta cos\alpha\hat{\boldsymbol{z}} + sin\theta cos\theta sin\alpha\hat{\boldsymbol{x}} + cos\theta sin\theta cos\alpha\hat{\boldsymbol{z}})$$

$$\boldsymbol{N}_m = -2m\omega r^2\dot{\theta}sin\alpha(sin\theta cos\theta\hat{\boldsymbol{x}} - cos^2\theta\hat{\boldsymbol{y}})$$

The differential mass is

$$dm = \lambda r d\theta$$

where λ is the linear mass density. The total torque is

$$\boldsymbol{N} = \int d\boldsymbol{N}_m$$

$$\boldsymbol{N} = -2\omega r^2\dot{\theta}sin\alpha\int(sin\theta cos\theta\hat{\boldsymbol{x}} - cos^2\theta\hat{\boldsymbol{y}})dm$$

$$\boldsymbol{N} = -2\omega r^3\lambda\dot{\theta}sin\alpha\int_0^{2\pi}\left(sin\theta cos\theta\hat{\boldsymbol{x}} - \frac{1}{2}(1 + cos2\theta)\hat{\boldsymbol{y}}\right)d\theta$$

$$\boldsymbol{N} = -2\omega r^3\lambda\dot{\theta}sin\alpha\left(\frac{sin^2\theta}{2}\bigg|_0^{2\pi}\hat{\boldsymbol{x}} - \frac{1}{2}\left(\theta + \frac{sin2\theta}{2}\right)\bigg|_0^{2\pi}\hat{\boldsymbol{y}}\right)$$

$$\boldsymbol{N} = -2\omega r^3\lambda\dot{\theta}sin\alpha\left(\frac{sin^2\theta}{2}\bigg|_0^{2\pi}\hat{\boldsymbol{x}} - \frac{1}{2}\left(\theta + \frac{sin2\theta}{2}\right)\bigg|_0^{2\pi}\hat{\boldsymbol{y}}\right)$$

$$\boldsymbol{N} = -2\omega r^3\lambda\dot{\theta}sin\alpha\left(0\hat{\boldsymbol{x}} - \frac{1}{2}(2\pi)\hat{\boldsymbol{y}}\right)$$

$$\boldsymbol{N} = 2\pi\omega r^3\lambda\dot{\theta}sin\alpha\hat{\boldsymbol{y}}$$

$$M = 2\pi r\lambda$$

$$\boldsymbol{N} = Mr^2\omega\dot{\theta}sin\alpha\hat{\boldsymbol{y}}$$

$$\hat{\boldsymbol{y}} = \boldsymbol{j}$$

$$\boxed{\boldsymbol{N} = \boldsymbol{j}Mr^2\omega\dot{\theta}sin\alpha}$$

This result does not agree with the given expression. A check is to consider the angular momentum of the gyroscope and determine its rate of change due to the earth's rotation. Using the same coordinate system,

$$\boldsymbol{L} = I\dot{\theta}\hat{\boldsymbol{z}}$$

$$I_{hoop} = Mr^2$$

$$\boldsymbol{L} = Mr^2\dot{\theta}\hat{\boldsymbol{z}}$$

$$\boldsymbol{\omega} = \omega(sin\alpha\hat{\boldsymbol{x}} + cos\alpha\hat{\boldsymbol{z}})$$

The relation between stationary and rotating derivatives is

$$\frac{d\boldsymbol{L}}{dt} = \frac{d^*\boldsymbol{L}}{dt} + \boldsymbol{\omega} \times \boldsymbol{L}$$

In the rotating frame the gyroscope is fixed. So,

$$\frac{d^*\boldsymbol{L}}{dt} = 0$$

$$\frac{d\boldsymbol{L}}{dt} = \omega(sin\alpha\hat{\boldsymbol{x}} + cos\alpha\hat{\boldsymbol{z}}) \times Mr^2\dot{\theta}\hat{\boldsymbol{z}}$$

$$\frac{d\boldsymbol{L}}{dt} = -Mr^2\omega\dot{\theta}sin\alpha\hat{\boldsymbol{y}}$$

This is the torque in the non-rotating coordinate system. Therefore, in the rotating frame the Coriolis (inertial) torque is

$$\frac{d^*\boldsymbol{L}}{dt} = Mr^2\omega\dot{\theta}sin\alpha\hat{\boldsymbol{y}}$$

It is noticed the moment of inertia for a solid disk is

$$I_{disk} = \frac{1}{2}Mr^2$$

and this is likely the reason for the stated relation. The reason for the sign difference is unclear and may have to do with the coordinate system orientation. It appears the stated problem did not exactly define the conditions necessary to arrive at the given expression for the torque.