8300 2003 Day 12, Tangent Spaces and Derivations

Just as a polynomial has a best linear approximation at each point, by its differential, so does an affine variety have a best linear approximation, by its tangent space. Naturally, the tangent space T_pX to a variety X at p, is defined by the best linear approximations at p of the polynomials defining X. There are several other useful descriptions of the tangent space. We will give 6 of them, beginning with a very simple intuitive geometric one, the union of all lines "touching" X at p.

Let X be a closed affine variety in A^m, and assume 0 is a point of X. Then any line through 0 must meet X at 0. I.e. if $v \neq 0$ is a non zero direction vector in A^m, and $g(t) = tv$ is a parametrization of the line L through 0 with direction v, and if f is any polynomial in the ideal I(X), then the "restriction" fog of f to L vanishes at t= 0. We say the line L touches X at 0, or is tangent to X at 0, if this line meets X "twice" at 0 in the following sense.

Definition: The line $L = \{g(t) = tv\}$, is tangent to X at 0 iff for all f in I(X), the composition fog vanishes at least twice at t=0, i.e. if and only if t^2 divides the polynomial (fog)(t).

If the point p on X is not 0, we apply the same condition to a parametrization of form L: $\{g(t) = p+tv\}$. Thus among all lines passing through p, we distinguish as tangent lines those that intersect X at p with multiplicity $\geq$ 2. In particular this includes all lines through p that are actually contained in X.

Preliminary Definition: If X is an affine subvariety of A^m, not necessarily irreducible, and p a point of X, the (embedded affine) tangent space T_pX to X at p, is the union of all lines through p in A^m which are tangent to X at p.

Next we want to show the tangent space T_pX is itself an affine algebraic subvariety of A^m by giving equations for it.

Definition: If f is a polynomial in $k[T_1,\dots,T_m]$, we define the differential df_p of f at a point $p = (p_1,\dots,p_m)$ of A^m to be the linear polynomial $df_p = \sum_{i=1,\dots,m} (\partial f/\partial T_i)(p)(T_i-p_i) = \sum_{i=1,\dots,m} D_if(p)(T_i-p_i)$, (homogeneous in T_i-p_i, but not in T_i) where the partial derivative of the polynomial f is defined by the usual "power" rule.

Remark: d_p is a linear map from $k[T_1,\dots,T_n]$ to itself, such that the fundamental rules of derivatives hold. I.e. d_p is a “derivation”.

(0) $d_p(c) = 0$, if c is in k.
(i) $d_p(f+g) = d_pf + d_pg$.
(ii) $d_p(fg) = f(p).d_pg + g(p).d_pf$.

Note: $\sum_{i=1,\ldots,m} D_if(p)(T_i-p_i)$ is a polynomial in two sets of variables (p,T), and which is linear in T but not necessarily in p.

Lemma: If X is an affine subvariety of A^m, and p a point of X, the tangent space T_pX is the common zero locus of the linear polynomials $\sum_{i=1,\ldots,m} D_if(p)(T_i-p_i) = d_pf$, for all f in I(X).
proof: For simplicity of notation we take p = 0. Then expand each polynomial f in I(X) as usual $f = f_1 + f_2 + \ldots+f_d$, where f_j is the homogeneous term of f of degree j. Then for any parametrized line g(t) = tv, the restriction (fog)(t) = $(f_1og)(t) + (f_2og)(t) + \ldots+ (f_dog)(t)$ is also decomposed into homogeneous terms, since $(f_jog)(t) = f_j(tv)$ is a monomial of degree j in t. Note that $f_1 = d_0f$, i.e. when p = 0, then $d_pf = f_1$, the homogeneous linear prt of f. Thus t^2 divides the terms $(f_2og)(t) + \ldots+ (f_dog)(t)$, and hence t^2 divides fog iff $(f_1og)(t)$ is identically zero, iff d_0f is identically zero on the line L = {g(t) = tv}.

It follows that T_0X is the union of all lines through 0 in the common zero locus of the differentials d_0f for all f in I(X). Since these polynomials are all linear, this common zero locus is a linear space, so it equals the union of the lines in it through 0. Hence T_0X is the common zero locus of the linear polynomials d_0f for all f in I(X). **QED.**

Exercise: Show it suffices in defining T_pX to use only the differentials of a set of generators $\{f_å\}$ for the ideal I(X).

Remark: It is important here that we use the full ideal I(X) of polynomials vanishing on X. I.e. although to define T_pX it is sufficient to take a set of generators f for I(X), it is not sufficient to use a non radical ideal with the same zero locus as I(X).

Example: **(i)** Consider the parabola X : $\{y-x^2 = 0\}$ in $!^2$. Then T_0X is defined by y = 0, so T_0X is the x - axis as expected.
(ii) If we use the polynomial $(y-x^2)^2 = 0$, which defines the same point set as the one above, but which is not square - free, the linear part of this polynomial

is zero, so the tangent space would appear to be all of A^2.

Later when we study schemes, and allow our schemes to be defined by non radical ideals, we will say that in this case the tangent space to the scheme defined by the ideal $((y-x^2)^2)$, at every point, is indeed all of A^2.

(iii) In the case of the cuspidal curve $\{y^2 = x^3\}$fi!^{2}, note that this time the tangent space at 0 in our sense is really all of !^{2}, but not at any other point. So for an affine plane curve defined by an irreducible polynomial f, the tangent space at a point p of the curve is one dimensional unless the gradient vector $(\partial f/\partial x(p), \partial f/\partial y(p)) = d_pf$ at that point is zero.

(iv) If X is a hypersurface in A^m, hence defined by a square free polynomial f, then at p on X, T_pX is defined by $d_pf = 0$. Thus the tangent space is either m dimensional if d_pf is identically zero, or m-1 dimensional otherwise. We claim the second case is more common.

Lemma: If $\{f=0\}$ is an irreducible equation for a hypersurface X in A^m then the tangent space T_pX, has dimension m-1 = dim(X) at least for a dense Zariski open subset of points p in X.

proof: The linear function defining the tangent space of X at p has coefficients given by the value at p of the partial derivatives of f. So we must show that at some point p of X, some partial derivative $D_if(p)$ of f does not vanish. In characteristic zero, if the degree of f is $d \geq 1$, then the degree of each partial derivative is $d-1 \geq 0$. Since I(X) = (f), the only polynomials that vanish identically on X are multiples of f, hence a partial derivative, being non zero and of lower degree than f, cannot be a multiple of f, hence cannot vanish identically on X. In characteristic $p > 0$, if all partials are multiples of f, since this time they either have degree d-1 or are zero, they must all be identically zero. Hence as argued before using the characteristic p binomial theorem, since k is algebraically closed, f would be a pth power, hence not irreducible, a contradiction. **QED.**

Corollary: If f is an irreducible polynomial in k[X,Y] and C:$\{f=0\}$ is the corresponding irreducible plane curve, there are only a finite number of "singular" points of C, i.e. points at which the gradient $(\partial f/\partial x, \partial f/\partial y)$ is zero. At all <u>non</u> singular points of C the tangent space of C is one dimensional, while at each singular point it is two dimensional.

Exercise:(i) If C: $\{y^2 = x^3+x^2\}$in A^2 show (0,0) is the only singular point of C. I.e. the tangent space there is 2 dimensional, and at every other point it is one dimensional.

(ii) If If S: $\{x^3+y^3+z^3+w^3 = 0\}$ in P^3, is the Fermat cubic surface and char(k) $\neq$ 3, then S has no singular points. E.g. in the open set $w \neq 0$, the affine equation is $\{x^3+y^3+z^3+1=0\}$, so the gradient $(3x^2,3y^2,3z^2)$ is zero only at (0,0,0) which is not a point of S.

We now have two equivalent definitions of the tangent space T_pX, but there are several other definitions, each having some utility, so we give six of them next. (Mumford gives two others, in terms of Kahler differentials, and "dual numbers", and one can also describe the tangent space in terms of the maximal ideal of the completion of the local ring.) Certainly we want Zariski's intrinsic definition 4), in terms of just the maximal ideal of the local ring of X at p. We also want to emphasize how the equivalence of the embedded and the intrinsic definitions depends on the existence of a "universal derivation".

Theorem: If X in A^n is a closed affine variety and p a point of X, the following definitions all define naturally isomorphic k - linear vector spaces, the "tangent space" to X at p.

1) $T_p(X)$ = the union of those lines in A^n which are tangent to X at p, i.e. which intersect X at p with multiplicity ≥ 2. (The linear structure of the vector space T_pX makes p the origin.)

2) $T_p(X) = \{q: d_pf(q) = 0$, all f in $I(X)\} = \{q$ in A^n such that for all f in I(X), $\sum_{i=1,\dots,n} D_if(p)(q_i-p_i) = 0\}$, the common zero locus of the linear terms d_pf, of the Taylor expansions at p of the elements f of the ideal I(X) of X in A^n. [Note that we take the linear terms of all elements in the full radical ideal I(X).]

3) $T_p(X) = \mathrm{Hom}_k(M/M^2, k) = (M/M^2)^*$, where M is the maximal ideal in k[X] of the point p. (This is intrinsic, in terms of the affine variety X, so isomorphic affine varieties have isomorphic tangent spaces at corresponding points.)

4) $T_p(X) = \mathrm{Hom}_k(m/m^2, k) = (m/m^2)^*$, where m is the maximal ideal in $O_{p,X}$ of the point p. (This definition is intrinsic and local on X, so the tangent space at p is determined by any neighborhood of p in X.)

5) $T_p(X) = \mathrm{Der}_p(k[X], k)$ = all k linear mappings D: k[X] -->k, which satisfy the Leibniz rule at p, i.e. such that D(fg) = f(p)D(g) + g(p)D(f), for all f,g in k[X]. (Intrinsic to X.)

6) $T_p(X) = \mathrm{Der}_p(O_{p,X}, k)$. (Intrinsic and local on X.)