Taylor expansion of metric

Posted on 16/03/2012 by KKK

This is an exercise in Riemannian geometry. In this note I Taylor-expand the Riemannian metric in a normal neighborhood around a point in a Riemannian manifold M. What I have done is nothing except providing a few more terms than most standard textbooks. Hopefully the computation is correct. (I haven't checked against other sources.)

> ***Theorem 1*** *In a normal coordinates neighborhood of $p\in M$, the Taylor series of g around p is given by*

$$\begin{array}{rcl} g_{ij}(x) &=& \delta_{ij}-\frac{1}{3}R_{iklj}x^kx^l-\frac{1}{6}R_{iklj;m}x^kx^lx^m\\ && +(\frac{2}{45}R_{ilmk}R_{jpqk}-\frac{1}{20}R_{ilmj;pq})x^lx^mx^px^q\\ && +(-\frac{1}{90}R_{iklj;mpq}+\frac{2}{45}R_{iklr;m}R_{jpqr})x^kx^lx^mx^px^q\\ && +(-\frac{1}{504}R_{iklj;mpqr}+\frac{17}{1260}R_{ikls;pq}R_{jmps}+\frac{11}{1008}R_{ikls;q}R_{jmps;r}\\ && +\frac{1}{315}R_{ilms}R_{jqrt}R_{kspt})x^kx^lx^mx^px^qx^r+O(|x|^7). \end{array}$$

Proof: In normal coordinates, fix $x=(x^1,\cdots,x^n)$ for the moment and let $\gamma=(tx^1,\cdots,tx^n)$ be a radial geodesic, then $J(t)=tW^i\partial_i$ is a Jacobi field on $\gamma(t)$, where $W^i\in\mathbb{R}$. (This can be seen by observing that tW is the variational vector field of the s-family of geodesics $\gamma_s(t)=\exp_p(t(x+sW))$, with a slight abuse of notations.) Let $f=\langle J,J\rangle$.

We will use $\cdot$ to denote a covariant derivative w.r.t. t and define $RX:=R(\dot\gamma,X)\dot\gamma$, then $\langle R(\dot\gamma,X)\dot\gamma,Y\rangle=\langle R(\dot\gamma,Y)\dot\gamma,X\rangle$ implies R is a symmetric operator. Differentiating $\langle RX,Y\rangle=\langle RY,X\rangle$ implies $\dot R,\ddot R,\dots$ are also symmetric. Also, the Jacobi equation simply becomes

$$\ddot J=RJ.$$

We compute (a way to test the correctness of the following computation is to note that each R carries two derivatives)

$$\begin{array}{rcl} f &=& \langle J,J\rangle.\\ f' &=& 2\langle \dot J,J\rangle.\\ f'' &=& 2\langle RJ,J\rangle+2\langle \dot J,\dot J\rangle.\\ f''' &=& 2\langle \dot RJ,J\rangle+8\langle R\dot J,J\rangle.\\ f^{(4)} &=& 2\langle \ddot RJ,J\rangle+(4+8)\langle \dot R\dot J,J\rangle+8\langle RRJ,J\rangle+8\langle R\dot J,\dot J\rangle\\ &=& 2\langle \ddot RJ,J\rangle+12\langle \dot R\dot J,J\rangle+8\langle RJ,RJ\rangle+8\langle R\dot J,\dot J\rangle \end{array}$$

$$\begin{array}{rcl} f^{(5)} &=& 2\langle \dddot RJ,J\rangle+(4+12)\langle \ddot R\dot J,J\rangle+12\langle \dot RRJ,J\rangle+(12+8)\langle \dot R\dot J,\dot J\rangle\\ && +16\langle \dot RJ,RJ\rangle+16\langle R\dot J,RJ\rangle+16\langle RRJ,\dot J\rangle\\ &=& 2\langle \dddot RJ,J\rangle+16\langle \ddot R\dot J,J\rangle+28\langle \dot RJ,RJ\rangle+20\langle \dot R\dot J,\dot J\rangle+32\langle R\dot J,RJ\rangle \end{array}$$

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$$
\begin{aligned}
f^{(6)} &= 2\langle \ddddot{R}J, J\rangle + (4+16)\langle \dddot{R}\dot{J}, J\rangle + 16\langle \ddot{R}RJ, J\rangle + (16+20)\langle \ddot{R}\dot{J}, \dot{J}\rangle + 28\langle \ddot{R}J, RJ\rangle \\
&\quad +(28+32)\langle \dot{R}\dot{J}, RJ\rangle + 28\langle \dot{R}J, \dot{R}J\rangle + (28+32)\langle \dot{R}J, R\dot{J}\rangle + 40\langle \dot{R}RJ, \dot{J}\rangle \\
&\quad +32\langle RRJ, RJ\rangle + 32\langle R\dot{J}, R\dot{J}\rangle \\
&= 2\langle \ddddot{R}J, J\rangle + 20\langle \dddot{R}\dot{J}, J\rangle + 16\langle \ddot{R}RJ, J\rangle + 36\langle \ddot{R}\dot{J}, \dot{J}\rangle + 28\langle \ddot{R}J, RJ\rangle + 60\langle \dot{R}\dot{J}, RJ\rangle \\
&\quad +28\langle \dot{R}J, \dot{R}J\rangle + 60\langle \dot{R}J, R\dot{J}\rangle + 40\langle \dot{R}RJ, \dot{J}\rangle + 32\langle RRJ, RJ\rangle + 32\langle R\dot{J}, R\dot{J}\rangle \\
&= 2\langle \ddddot{R}J, J\rangle + 20\langle \dddot{R}\dot{J}, J\rangle + 44\langle \ddot{R}J, RJ\rangle + 36\langle \ddot{R}\dot{J}, \dot{J}\rangle + 100\langle \dot{R}\dot{J}, RJ\rangle + 28\langle \dot{R}J, \dot{R}J\rangle \\
&\quad +60\langle \dot{R}J, R\dot{J}\rangle + 32\langle RRJ, RJ\rangle + 32\langle R\dot{J}, R\dot{J}\rangle
\end{aligned}
$$

$$
\begin{aligned}
f^{(7)} &= 2\langle R^{(5)}J, J\rangle + (4+20)\langle \ddddot{R}\dot{J}, J\rangle + 20\langle \dddot{R}RJ, J\rangle + (20+36)\langle \dddot{R}\dot{J}, \dot{J}\rangle + 44\langle \dddot{R}J, RJ\rangle \\
&\quad +(44+100)\langle \ddot{R}\dot{J}, RJ\rangle + (44+56)\langle \ddot{R}J, \dot{R}J\rangle + (44+60)\langle \ddot{R}J, R\dot{J}\rangle + 72\langle \ddot{R}RJ, \dot{J}\rangle \\
&\quad +(100+32)\langle \dot{R}RJ, RJ\rangle + (100+56+60)\langle \dot{R}\dot{J}, \dot{R}J\rangle + (100+60+64)\langle \dot{R}\dot{J}, R\dot{J}\rangle \\
&\quad +60\langle \dot{R}J, RRJ\rangle + 64\langle R\dot{R}J, RJ\rangle + 64\langle RR\dot{J}, RJ\rangle + 64\langle RRJ, R\dot{J}\rangle \\
&= 2\langle R^{(5)}J, J\rangle + 24\langle \ddddot{R}\dot{J}, J\rangle + 20\langle \dddot{R}RJ, J\rangle + 56\langle \dddot{R}\dot{J}, \dot{J}\rangle + 44\langle \dddot{R}J, RJ\rangle + 144\langle \ddot{R}\dot{J}, RJ\rangle \\
&\quad +100\langle \ddot{R}J, \dot{R}J\rangle + 104\langle \ddot{R}J, R\dot{J}\rangle + 72\langle \ddot{R}RJ, \dot{J}\rangle + 132\langle \dot{R}RJ, RJ\rangle + (216\langle \dot{R}\dot{J}, \dot{R}J\rangle \\
&\quad +224\langle \dot{R}\dot{J}, R\dot{J}\rangle + 60\langle \dot{R}J, RRJ\rangle + 64\langle R\dot{R}J, RJ\rangle + 64\langle RR\dot{J}, RJ\rangle + 64\langle RRJ, R\dot{J}\rangle \\
&= 2\langle R^{(5)}J, J\rangle + 24\langle \ddddot{R}\dot{J}, J\rangle + 20\langle \dddot{R}RJ, J\rangle + 56\langle \dddot{R}\dot{J}, \dot{J}\rangle + 44\langle \dddot{R}J, RJ\rangle + 216\langle \ddot{R}\dot{J}, RJ\rangle \\
&\quad +100\langle \ddot{R}J, \dot{R}J\rangle + 104\langle \ddot{R}J, R\dot{J}\rangle + 132\langle \dot{R}RJ, RJ\rangle + (216\langle \dot{R}\dot{J}, \dot{R}J\rangle + 224\langle \dot{R}\dot{J}, R\dot{J}\rangle \\
&\quad +60\langle \dot{R}J, RRJ\rangle + 64\langle R\dot{R}J, RJ\rangle + 128\langle RR\dot{J}, RJ\rangle
\end{aligned}
$$

(Actually to compute $f^{(i)}(0)$ we don't have to compute all the terms of $f^{(i)}(t)$: we can omit the term containing any $J = J(0) = 0$). It follows that

$$
\begin{aligned}
f(0) &= 0. \\
f'(0) &= 0. \\
f''(0) &= \langle W, W\rangle. \\
f'''(0) &= 0. \\
f^{(4)}(0) &= 8\langle W, RW\rangle = -8\langle R(W,\dot{\gamma})\dot{\gamma}, W\rangle. \\
f^{(5)}(0) &= 20\langle W, \dot{R}W\rangle = -20\langle \nabla_t R(\dot{\gamma}, W)W, \dot{\gamma}\rangle. \\
f^{(6)}(0) &= 32\langle RW, RW\rangle + 36\langle W, \ddot{R}W\rangle = 32\langle R(W,\dot{\gamma},)\dot{\gamma}, R(W,\dot{\gamma})\dot{\gamma}\rangle - 36\langle \nabla_t^2 R(W,\dot{\gamma})\dot{\gamma}, W\rangle. \\
f^{(7)}(0) &= 56\langle \dddot{R}W, W\rangle + 224\langle \dot{R}W, RW\rangle \\
&= -56\langle \nabla_t^3 R(W,\dot{\gamma})\dot{\gamma}, W\rangle + 224\langle \nabla_t R(W,\dot{\gamma})\dot{\gamma}, R(W,\dot{\gamma})\dot{\gamma}\rangle. \\
f^{(8)}(0) &= (24+56)\langle \ddddot{R}W, W\rangle + (216+104+224)\langle \ddot{R}W, RW\rangle + (216+224)\langle \dot{R}W, \dot{R}W\rangle \\
&\quad +128\langle RRW, RW\rangle \\
&= 80\langle \ddddot{R}W, W\rangle + 544\langle \ddot{R}W, RW\rangle + 440\langle \dot{R}W, \dot{R}W\rangle + 128\langle RRW, RW\rangle
\end{aligned}
$$

All the above expression are evaluated at 0, noting that $J(0) = 0$. So we have (all repeated indices are summed over):

$$
\begin{aligned}
\boxed{f(t) = t^2 g_{ij}(tx) W^i W^j} \\
&= \sum_{i=0} \frac{1}{i!} f^{(i)}(0) t^i + O(t^8) \\
&= \delta_{ij} t^2 W^i W^j - \tfrac{1}{3} R_{iklj} x^k x^l t^4 W^i W^j - \tfrac{1}{6} R_{iklj;m} x^k x^l x^m t^5 W^i W^j \\
&\quad +(\tfrac{2}{45} R_{ilmk} R_{jpqk} - \tfrac{1}{20} R_{ilmj;pq}) x^l x^m x^p x^q t^6 W^i W^j \\
&\quad +(-\tfrac{1}{90} R_{iklj;mpq} + \tfrac{2}{45} R_{iklr;m} R_{jpqr}) x^k x^l x^m x^p x^q t^7 W^i W^j \\
&\quad +(-\tfrac{1}{504} R_{iklj;mpqr} + \tfrac{17}{1260} R_{ikls;pq} R_{jmps} + \tfrac{11}{1008} R_{ikls;q} R_{jmps;r} \\
&\quad +\tfrac{1}{315} R_{ilms} R_{jqrt} R_{kspt}) x^k x^l x^m x^p x^q x^r t^8 W^i W^j + O(t^9).
\end{aligned}
$$

We thus conclude that

$$
\begin{aligned}
g_{ij}(x) &= \delta_{ij} - \tfrac{1}{3} R_{iklj} x^k x^l - \tfrac{1}{6} R_{iklj;m} x^k x^l x^m \\
&\quad +(\tfrac{2}{45} R_{ilmk} R_{jpqk} - \tfrac{1}{20} R_{ilmj;pq}) x^l x^m x^p x^q \\
&\quad +(-\tfrac{1}{90} R_{iklj;mpq} + \tfrac{2}{45} R_{iklr;m} R_{jpqr}) x^k x^l x^m x^p x^q \\
&\quad +(-\tfrac{1}{504} R_{iklj;mpqr} + \tfrac{17}{1260} R_{ikls;pq} R_{jmps} + \tfrac{11}{1008} R_{ikls;q} R_{jmps;r} \\
&\quad +\tfrac{1}{315} R_{ilms} R_{jqrt} R_{kspt}) x^k x^l x^m x^p x^q x^r + O(|x|^7).
\end{aligned}
$$

Here $|x| = r$ is the radial distance from p. □

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5 Responses to *Taylor expansion of metric*

Ken Leung *says:*
17/03/2012 at 5:58 am

The lengths of the formulae are remarkable. Thanks for KKK's hard work in typesetting them.

Reply

Salem *says:*
25/08/2012 at 9:20 pm

in other places (for example http://arxiv.org/abs/1206.1092) the O(x^2) term has the opposite sign to the result here (j and l exchanged)

Reply

KKK *says:*
27/08/2012 at 4:23 pm

While I haven't checked the higher order terms, the terms up to the forth order are correct, see e.g. Hamilton's Ricci Flow p.59. The reason for the difference is that I have used a different convention for the Riemannian curvature tensor (my apology for not stating it): my definition is $R_{ijkl} = \langle (\nabla_{e_i}\nabla_{e_j} - \nabla_{e_j}\nabla_{e_i} - \nabla_{[e_i,e_j]})e_k, e_l \rangle$.

Reply

Salem *says:*

28/08/2012 at 3:00 am

Thanks!

Reply

zidane *says:*

29/05/2015 at 9:59 pm

which can answer to my question: how to calculate the inverse of the metric g_ {ij}, in other words how to express g ^ {ij} -Thank you

Reply

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