Proof of a Tensor Identity

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I. Introduction

THE problem is the verification of a tensor identity with the *indicial notation*(no other method should be used).

II. Curl of the Transpose of a Gradient of a Vector

The identity that has to be proven is the following:

$$\text{curl}\left(\boldsymbol{\nabla}\boldsymbol{u}^T\right) = \boldsymbol{\nabla}\text{curl}\boldsymbol{u} \tag{1}$$

or, in an equivalent writing:

$$\boldsymbol{\nabla}\times\left(\boldsymbol{\nabla}\boldsymbol{u}^T\right) = \boldsymbol{\nabla}\left(\boldsymbol{\nabla}\times\boldsymbol{u}\right) \tag{2}$$

My attempt follows. Please find where I make the mistake!
The vector in indicial notation is:

$$\boldsymbol{u} = u_i\widehat{e}_i \tag{3}$$

The operator $\boldsymbol{\nabla}$ is:

$$\boldsymbol{\nabla} = \partial_i\widehat{e}_i \tag{4}$$

The gradient of vector $\boldsymbol{u}$ is the following *tensor*:

$$\boldsymbol{\nabla}\boldsymbol{u} = \partial_i\widehat{e}_i u_j\widehat{e}_j = u_{j,i}\,\widehat{e}_i\widehat{e}_j \tag{5}$$

where $u_{j,i} = \frac{\partial u_j}{\partial x_i}$. The transpose of the gradient of $\boldsymbol{u}$ is obtained from 5:

$$\boldsymbol{\nabla}\boldsymbol{u}^T = u_{j,i}\,\widehat{e}_j\widehat{e}_i \equiv u_{i,j}\,\widehat{e}_i\widehat{e}_j \tag{6}$$

The curl of the transpose of the gradient of a vector is (see equations 4 and 6):

$$\boldsymbol{\nabla}\times\left(\boldsymbol{\nabla}\boldsymbol{u}^T\right) = \partial_k\widehat{e}_k\times u_{i,j}\,\widehat{e}_i\widehat{e}_j = \left(u_{i,jk}\widehat{e}_k\times\widehat{e}_i\right)\widehat{e}_j = \varepsilon_{kip}u_{i,jk}\widehat{e}_p\widehat{e}_j \equiv \varepsilon_{kip}u_{i,kj}\widehat{e}_p\widehat{e}_j \tag{7}$$

Where ε_{kip} is the permutation symbol.
Now we work on the RHS of equation 2.

$$\boldsymbol{\nabla}\left(\boldsymbol{\nabla}\times\boldsymbol{u}\right) = \partial_p\widehat{e}_p\left(\partial_i\widehat{e}_i\times u_j\widehat{e}_j\right) = \partial_p\widehat{e}_p\left(u_{j,i}\widehat{e}_i\times\widehat{e}_j\right) = \partial_p\widehat{e}_p\left(\varepsilon_{ijk}u_{j,i}\widehat{e}_k\right) = \varepsilon_{ijk}u_{j,ip}\widehat{e}_p\widehat{e}_k \equiv \varepsilon_{ijk}u_{j,pi}\widehat{e}_p\widehat{e}_k \tag{8}$$

If the identity represented by equation 2 is true, then equations 8 and 7 must represent the same quantity. However, it seems that even if I change the symbols and I use the permutation symbol's properties I can not demonstrate that. Could you please help me and post your derivation (please post only the indicial notation)?