



The Laplacian Operator



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The Laplacian Operator from Cartesian to Cylindrical to Spherical Coordinates

The Laplacian Operator is very important in physics. It is nearly ubiquitous. Its form is simple and symmetric in Cartesian coordinates.

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Before going through the Carpal-Tunnel causing calisthenics to calculate its form in cylindrical and spherical coordinates, the results appear here so that more intelligent people can just move along without troubling themselves. In cylindrical form:

$$\nabla^2 = \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}$$

In spherical coordinates:

$$\nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2}$$

Converting to Cylindrical Coordinates

The painful details of calculating its form in cylindrical and spherical coordinates follow. It is good to begin with the simpler case, cylindrical coordinates. The z component does not change. For the x and y components, the transformations are

$$\begin{aligned} x &= \rho \cos \phi \\ y &= \rho \sin \phi \end{aligned}$$

inversely,

$$\begin{aligned} \rho &= \sqrt{x^2 + y^2} \\ \cos \phi &= \frac{x}{\sqrt{x^2 + y^2}} \\ \sin \phi &= \frac{y}{\sqrt{x^2 + y^2}} \end{aligned}$$

The chain rule relates the Cartesian operators to the cylindrical operators:

$$\frac{\partial}{\partial x} = \frac{\partial \rho}{\partial x} \frac{\partial}{\partial \rho} + \frac{\partial \phi}{\partial x} \frac{\partial}{\partial \phi}$$

and

$$\frac{\partial}{\partial y} = \frac{\partial \rho}{\partial y} \frac{\partial}{\partial \rho} + \frac{\partial \phi}{\partial y} \frac{\partial}{\partial \phi}$$

Calculate the derivatives for the chain rule. First ρ :

$$\frac{\partial \rho}{\partial x} = \frac{\partial}{\partial x} \sqrt{x^2 + y^2} = \frac{x}{\sqrt{x^2 + y^2}} = \frac{x}{\rho} = \cos \phi$$

$$\frac{\partial \rho}{\partial y} = \frac{\partial}{\partial y} \sqrt{x^2 + y^2} = \frac{y}{\sqrt{x^2 + y^2}} = \frac{y}{\rho} = \sin \phi$$

Now ϕ , using implicit differentiation:

$$-\sin \phi \, d\phi = \left(\frac{1}{\sqrt{x^2 + y^2}} - \frac{x^2}{(x^2 + y^2)^{3/2}} \right) dx = \frac{\sin^2 \phi}{\rho} dx$$

$$\implies \frac{\partial \phi}{\partial x} = -\frac{\sin \phi}{\rho}$$

$$\cos \phi \, d\phi = \left(\frac{1}{\sqrt{x^2 + y^2}} - \frac{y^2}{(x^2 + y^2)^{3/2}} \right) dy = \frac{\cos^2 \phi}{\rho} dy$$

$$\implies \frac{\partial \phi}{\partial y} = \frac{\cos \phi}{\rho}$$

Now write the Cartesian derivatives in terms of the cylindrical derivatives.

$$\frac{\partial^2}{\partial x^2} = \left(\cos \phi \frac{\partial}{\partial \rho} - \frac{\sin \phi}{\rho} \frac{\partial}{\partial \phi} \right) \left(\cos \phi \frac{\partial}{\partial \rho} - \frac{\sin \phi}{\rho} \frac{\partial}{\partial \phi} \right)$$

$$\frac{\partial^2}{\partial y^2} = \left(\sin \phi \frac{\partial}{\partial \rho} + \frac{\cos \phi}{\rho} \frac{\partial}{\partial \phi} \right) \left(\sin \phi \frac{\partial}{\partial \rho} + \frac{\cos \phi}{\rho} \frac{\partial}{\partial \phi} \right)$$

Think FOIL. $F_x + F_y$: the sum of the products of the first terms for the two derivatives gives a second derivative with respect to ρ . $O_x + O_y$: the outside terms cancel. $I_x + I_y$: the sum of the inside terms gives the derivative with respect to ρ divided by ρ . $L_x + L_y$: the sum of the products of the last terms for the two derivatives gives a second derivative with respect to ϕ divided by ρ^2 . Put it all together to get the Laplacian in cylindrical coordinates.

$$\nabla^2 = \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}$$

Converting to Spherical Coordinates

Now for the dreaded conversion to spherical coordinates. Before getting lost in this messy morass, perhaps it's good to remember why we care. We care because fundamental forces are believed to act directly between particles, with intermediate particles being the carriers of forces; mathematically, this means that these forces act radially. The gravitational force between masses and the electric force between charged particles are the two most common examples. So everything becomes much simpler if the angular parts can be resolved on their own. Spherical coordinates are the natural basis for this separation in three dimensions. The transformations of the coordinates themselves look rather innocuous. From spherical to Cartesian:

$$\begin{aligned} x &= r \sin \theta \cos \phi \\ y &= r \sin \theta \sin \phi \\ z &= r \cos \theta \end{aligned}$$

Or from Cartesian to spherical:

$$\begin{aligned} r &= \sqrt{x^2 + y^2 + z^2} \\ \cos \theta &= \frac{z}{\sqrt{x^2 + y^2 + z^2}} & \cos \phi &= \frac{x}{\sqrt{x^2 + y^2}} \\ \sin \theta &= \frac{\sqrt{x^2 + y^2}}{\sqrt{x^2 + y^2 + z^2}} & \sin \phi &= \frac{y}{\sqrt{x^2 + y^2}} \end{aligned}$$

Now take derivatives. Begin with r .

$$\frac{\partial r}{\partial x} = \frac{x}{r} = \sin \theta \cos \phi$$

$$\frac{\partial r}{\partial y} = \frac{y}{r} = \sin \theta \sin \phi$$

$$\frac{\partial r}{\partial z} = \frac{z}{r} = \cos \theta$$

Next find the partial derivatives of θ , once again by implicit differentiation. To find the derivatives with respect to x and y , use the expression for $\cos(\theta)$.

$$-\sin \theta \, d\theta = \frac{z(-x)}{r^3} dx \implies \frac{\partial \theta}{\partial x} = \frac{\cos \theta \cos \phi}{r}$$

$$-\sin \theta \, d\theta = \frac{z(-y)}{r^3} dy \implies \frac{\partial \theta}{\partial y} = \frac{\cos \theta \sin \phi}{r}$$

Then use the expression for $\sin(\theta)$ to find the derivative with respect to z .

$$\cos \theta \, d\theta = \frac{\sqrt{x^2 + y^2}(-z)}{r^3} dz \implies \frac{\partial \theta}{\partial z} = -\frac{\sin \theta}{r}$$

The derivatives of ϕ are the same as they were for cylindrical coordinates. For posterity, here they are in terms of spherical coordinates.

$$\frac{\partial \phi}{\partial x} = -\frac{\sin \phi}{r \sin \theta} \quad \frac{\partial \phi}{\partial y} = \frac{\cos \phi}{r \sin \theta}$$

Now comes the chain rule. This time, it's a bit uglier, since there are three variables involved. The simplest of the three terms in the Cartesian Laplacian to translate is z , since it is independent of the azimuthal angle.

$$\frac{\partial^2}{\partial z^2} = \left(\cos \theta \frac{\partial}{\partial r} - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \right) \left(\cos \theta \frac{\partial}{\partial r} - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \right)$$

The x and y versions are rather abominable.

$$\begin{aligned} \frac{\partial^2}{\partial x^2} &= \left(\sin \theta \cos \phi \frac{\partial}{\partial r} + \frac{\cos \theta \cos \phi}{r} \frac{\partial}{\partial \theta} - \frac{\sin \phi}{r \sin \theta} \frac{\partial}{\partial \phi} \right) \\ &\quad \times \left(\sin \theta \cos \phi \frac{\partial}{\partial r} + \frac{\cos \theta \cos \phi}{r} \frac{\partial}{\partial \theta} - \frac{\sin \phi}{r \sin \theta} \frac{\partial}{\partial \phi} \right) \end{aligned}$$

$$\begin{aligned} \frac{\partial^2}{\partial y^2} &= \left(\sin \theta \sin \phi \frac{\partial}{\partial r} + \frac{\cos \theta \sin \phi}{r} \frac{\partial}{\partial \theta} + \frac{\cos \phi}{r \sin \theta} \frac{\partial}{\partial \phi} \right) \\ &\quad \times \left(\sin \theta \sin \phi \frac{\partial}{\partial r} + \frac{\cos \theta \sin \phi}{r} \frac{\partial}{\partial \theta} + \frac{\cos \phi}{r \sin \theta} \frac{\partial}{\partial \phi} \right) \end{aligned}$$

This calls for an organized approach. All told, there is a total of 22 terms. First consider those that go like the second derivative of r . Collecting the three terms (one x , one y , one z), to find that they contribute

$$[\cos^2 \theta + \sin^2 \theta (\cos^2 \phi + \sin^2 \phi)] \frac{\partial^2}{\partial r^2} = \frac{\partial^2}{\partial r^2}$$

That wasn't so bad. Three down, 19 to go. Clinging to the fringes of the equations, next consider the terms that go like the second derivative of ϕ and find the contribution

$$\frac{\sin^2 \phi - \sin \phi \cos \phi + \cos^2 \phi + \cos \phi \sin \phi}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2} = \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2}$$

The result is not quite as pretty, but it was still mostly painless. Only 17 terms remain! To complete the diagonal derivatives, gather the three terms that come from the product of two derivatives with respect to θ to find the contribution

$$\frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}$$

Some easy math has been omitted. That was another three terms, which leaves 14. For the next bout of sadomasochism, collect all terms that result from the product of an r derivative and a θ derivative. This way, at least all z terms will be finished. The contribution is

$$\frac{\sin^2 \theta + \cos^2 \theta}{r} \frac{\partial}{\partial r} = \frac{1}{r} \frac{\partial}{\partial r}$$

If you didn't notice that some serious cancellation of cross terms happened before that pretty equation could be written, then you're not really paying attention. If you did notice, then you probably also noticed that 6 terms from our 22 term torture session just bit the dust.

Consider the 4 θ - ϕ cross terms. When the dust settles, the remaining bit is

$$\frac{(\sin^2 \phi + \cos^2 \phi) \cos \theta}{r^2 \sin \theta} \frac{\partial}{\partial \theta} = \frac{\cos \theta}{r^2 \sin \theta} \frac{\partial}{\partial \theta}$$

Only four more terms! Eager to see the end, tally the contributions from r - ϕ cross terms. The terms that don't cancel each other give the contribution

$$\frac{\sin^2 \phi \sin \theta + \cos^2 \phi \sin \theta}{r \sin \theta} \frac{\partial}{\partial r} = \frac{1}{r} \frac{\partial}{\partial r}$$

Now we gather all the terms to write the Laplacian operator in spherical coordinates:

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\cos \theta}{r^2 \sin \theta} \frac{\partial}{\partial \theta} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2}$$

This can be rewritten in a slightly tidier form:

$$\nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2}$$

Notice that multiplying the whole operator by r^2 completely separates the angular terms from the radial term. That is why all that work was worthwhile. Now it's time to solve some partial differential equations!!! See Legendre Polynomials and Spherical Harmonics.

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