CALCULUS -A UNIFICATION OF LINEAR AND NON-LINEAR MATHEMATICS

Art students learn Art History to have a sense of appreciation for it's development and to feel more a part of their chosen discipline.
Mathematics has a colourful history but it's beauty lies in it's precision.
Calculus is particularly rich in intricacy and was developed from the surprising formulation of non-linearity in linear terms.

Instead of looking at this subject from a historical perspective, bringing in the personalities involved, let's look at it from the conversational viewpoint of a student curious about what it's all about and a teacher determined to inspire him to find his own understanding of it, so that he learns with a sense of accomplishment and appreciation.

Children learn at a rapid rate while exploring their new world. They love stories because they are curious and have wonder. That may begin to diminish over time as they are exposed to education that chooses to ignore such joy of learning, especially when the instructors chose their profession for their own security.

To learn, we explore really accurately. It's also possible to be indoctrinated, whereby we can become "mentally cloned", programmed, as our brains are cybernetic, computer-like so it pays to do a regular "software check" on that!

To learn calculus, the spirit of exploration will lead you to full comprehension.
Learning the truth about things is fun. Repeating what the book says is not so much fun unless it's totally accurate and you see that it is so. Richard Feynman always taught his subject with immense enthusiasm because he loved it and enjoyed sharing it with his serious students. There is great joy in trying to help a student reach an even higher understanding than the facilitator.
http://inside.mines.edu/~tfurtak/feb14_2001/transcript.html

A: "What are you studying?"
B: Calculus.
A: How are you getting on with it?
B: I'm finding these limits hard to work with at times, I'm just trying to learn what I can of them, then I'm off to my friend's boxing match later.
A: Do you like boxing?
B: Sure, it's a tough one tonight, two very different styles and evenly matched on form.
A: Then you should find understanding limits a breeze, they are just like that.
B: You're kidding!
A: No, honestly, you're interested in boxing, right? I'm not, really, but you are.
B: Absolutely.
A: And I take it, your math class is a chore.
B: You know I'd rather be at the ring, it's way more exciting.
A: I can imagine! But Calculus is far more interesting than any textbook gives credit to it. Do you want to see what I mean?
B: Sure.

A: As you are following these limits to the limit, can you see what they look like?
B: It depends on the type of limit, some approach a line and the value is easy to see, like

$$\lim_{x \to \infty} \left(\frac{1}{x} \right) = 0$$ that's simple to see because of the ever-increasing relative magnitude of the

denominator to the numerator, but derivatives are unusual sometimes.
A: There is a way to see those but I recommend you use a pair of limits to understand it better.

One limit approaches zero, but the "clarifying limit" can start at "close to zero" and you can magnify it to any size you want, like a scientist using a microscope. The limit that's going to zero sometimes gives students headaches because they can't always imagine it's final shape. But the one you can peer at through the microscope never changes shape and is in fact the other awkward one at the limit. Therefore you can view the exact limit. How does that sound?

B: We can see the limit? I thought the wizardry of Calculus worked that out!

A: No! It merely "formulates the vision" in mathematically accurate terms.
It's brilliant, it really is. So is "Proof by Induction" but I haven't seen that get the credit it deserves either.

B: Sounds good, well I'd like to see how that works. What textbook is this from?

A: I honestly don't know.

B: Ok, anyway I'm out of time, gotta go see the fight, see you tomorrow.

(Next day)

A: How'd the fight go?

B: Great! He won on points.

A: Good, now's your turn to triumph over the perplexities of Calculus! Your mind is predispositioned to victory, success and perseverence to the end, you are in the right frame of mind to master this, with my assistance but not my instruction since a teacher is at best only a learning facilitator or if he's not serious, a learning hindrance. I have no intent to hinder you.

B: You mean you can't teach me.

A: Anything that's being taught depends totally on how the student chooses to receive it. I will show you what I see and you can examine how that feels to you. In truth, anyone that ever was really good at anything was mostly self-taught or worked hard to determine for themselves how good their teacher was at the subject, though that's really cutting a long story short. Ok, let's look at the Calculus.

Calculus was born of a straight line segment that had only one purpose.. TO DISAPPEAR!

B: That's really odd!

A: But true!

B: Seriously?

A: Yes, it's very true. It revolves around the fact that we make measurements using the shortest distance between two points which is a straight line.
I want you to consider this:
We use millimeters, centimeters, kilometres, seconds, hours and so on to measure.
These are all linear. Suppose we invented non-linear units to perform measurements with, what would happen?

B: We'd need one type of unit for circles, another for ellipses, wait...ellipses can have infinitely different shapes!! well, they are oval but you know what I mean.

A: Not to mention the shapes of infinitely varied polynomials, trigonometric curves etc.

B: Yes, and there is only ONE TYPE OF STRAIGHT LINE opposed to infinitely varied curves!

A: Yes, so we stick with the straight line otherwise we have a disorganised chaos in trying to make measurements.

B: In using straight lines, we have "organised chaos".

A: That's a cute way to put it! Now the unfortunate thing about using straight lines is that they give us inaccurate results when dealing with curvature.

B: Unless we can make them disappear.

A: Which is.....

B: CALCULUS!!

A: We are looking at the background of the development of Calculus to be able to see how Calculus developed and also how we as human beings bring workable structure to "apparently unworkable" non-linearity.

Underlying our efforts is the humble straight line which we are prepared to sacrifice once it's performed it's role like a kamikaze pilot. Sorry for sounding dramatic!

If we are going to start out with lines, it would help to fully understand them.

B: But lines are really straightforward, aren't they?

A: Do you remember what the weapons designer said in the film "Under Siege 2" about "assumptions"?

B: Yes, I saw that!

A: Remember I said, we would destroy the straight line after using it?

B: Yes.

A: Well, after destroying it, we will have a complete understanding of derivatives by observing that line resurrect itself and become indestructible!!

Linear is "the simplest way to look".

We measure time in a linear way, we think in time so our thought is linear, at least our conscious thought is, therefore we approach non-linearity in a linear way.

In the mathematics of Calculus, we begin to transcend our linear view but we approach it first in a linear way (first principles).

This linear mode is our conditioning, it is not original, it is secondary. However, it's advantage in Calculus is that it allows us to formulate complexity in simple manageable ways and that's where it deserves appreciation.

So you see, if you really want to have a feel for Calculus, it helps to be aware how we look at life itself, how we think and make sense of our environment.

B: I didn't realise it was so philosophical.

A: It's not really, we're just having a laugh while having a clear view of this incredible subject, how it was born so we can understand or track it's development or unfoldment.

We are setting the scene, so to speak, so as not to take Calculus for granted, to give it it's due respect.

Many feel Calculus is an intuitive branch of mathematics. It's not, it's the analysis of non-linear mathematics approached in a linear way through revelation of limits.

Now, to fully understand the humble line, lets look a picture of one.

Take a look at fig1, the visual representation of a part of "2 times" multiplication, or y=2x for all real values of x.

At any point (x,y) on the line, the value of y will be 2x, no matter what x is.

For the line's relevance in Calculus, I want you to focus on the large pink triangle.

The ratio of it's perpendicular sides is $\dfrac{\Delta y}{\Delta x}$ and this ratio is always 2 for this line no matter what size the triangle is, no matter where the triangle is.

You have no trouble imagining this triangle approach zero size as it will never change shape, the side ratios will always be 2, you can imagine shrinking yourself until you are smaller than the triangle even if you shrunk the triangle to atomic levels. Just like infinity, once you are nearing it, you are nowhere near it, the same with approaching zero, you can still imagine it 100 times smaller again and again.

This is just one piece of the puzzle. Now you can think of the line in 2 ways though they are not seperate.

B: Ok, I've got all that, I thought you said it was like boxing?

*fig*1

$y = 2x$
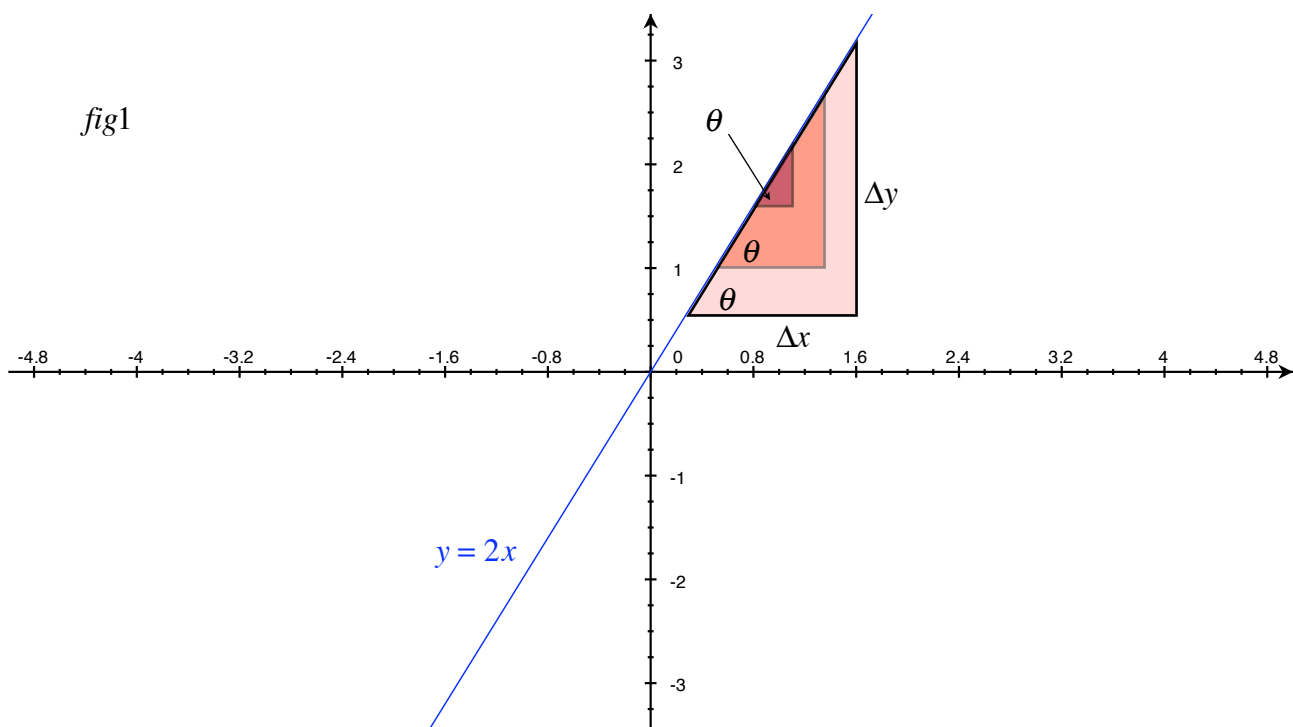
A: Ok, we've introduced the triangle from the blue corner, now let's introduce the red corner triangle.

B: Cool!

A: In the red corner, we have the "culprit" that is responsible for the $\lim\limits_{\Delta x \to 0}$

The way we resolve confusion and contention is through understanding, it's very helpful to know why we have these two triangles!

There is still contention about the symbolism of Calculus and what we are doing here is fully exploring this so that you can quickly tear through learning the remainder of all you want to learn about Calculus unhindered.

Imagine you are driving in your car, you check your speed on the dial and read your speed in real time. The tachometer operates as fast as possible to appear that it's measuring our speed without a time delay, but there is a little delay but to us milliseconds, microseconds etc are not critical, not under normal driving conditions. Admittedly, driving is a crude example but it's something everyone can relate to.

Mathematically, how would we calculate speed in a basic sense? We measure the time taken to cover a short distance and calculate distance/time.

Unfortunately, this averages the speed over that time interval and no matter what the time interval is, it will always be an average reading.

At this stage, realise that in fig1, we can write the line gradient using any size Δx and Δy since their ratio is fixed for this line.

We will only have instantaneous speed if the time interval is zero but we cannot divide by that! Still, you are aware that being exact gives you NO TIME TO COMPUTE THE RATIO. We resolve this by looking at the graph of distance versus time and noticing that at any point on the graph, the tangent gradient is it's instantaneous rate of change.

B: So, at the "zero width" point of tangency, we can magnify that "point" to any size we want and choose it's "shape" to be that triangle, inside of a circle or any other shape!!

A: That's it.

B: It's obvious when you think about it, but can we really choose the shape?

A: So think, and don't let your thinking be done for you, or rather...open your mind's eye.

As we are computing derivatives, this right-angled triangle is the shape that concerns us. Any shape you want exists there, as silly as that sounds!

B: That's it then!
A: How do you write the mathematics with the tangent?
B: The derivative is the tangent gradient.
A: How do you calculate it?
B: From two points on the tangent.
A: Yes, but we've only got one!!
B: Gotcha! We're stuck!
A: Exactly. So... enter the red corner triangle to give us a second point.
B: But that point won't be on the tangent, it'll give us a wrong answer.
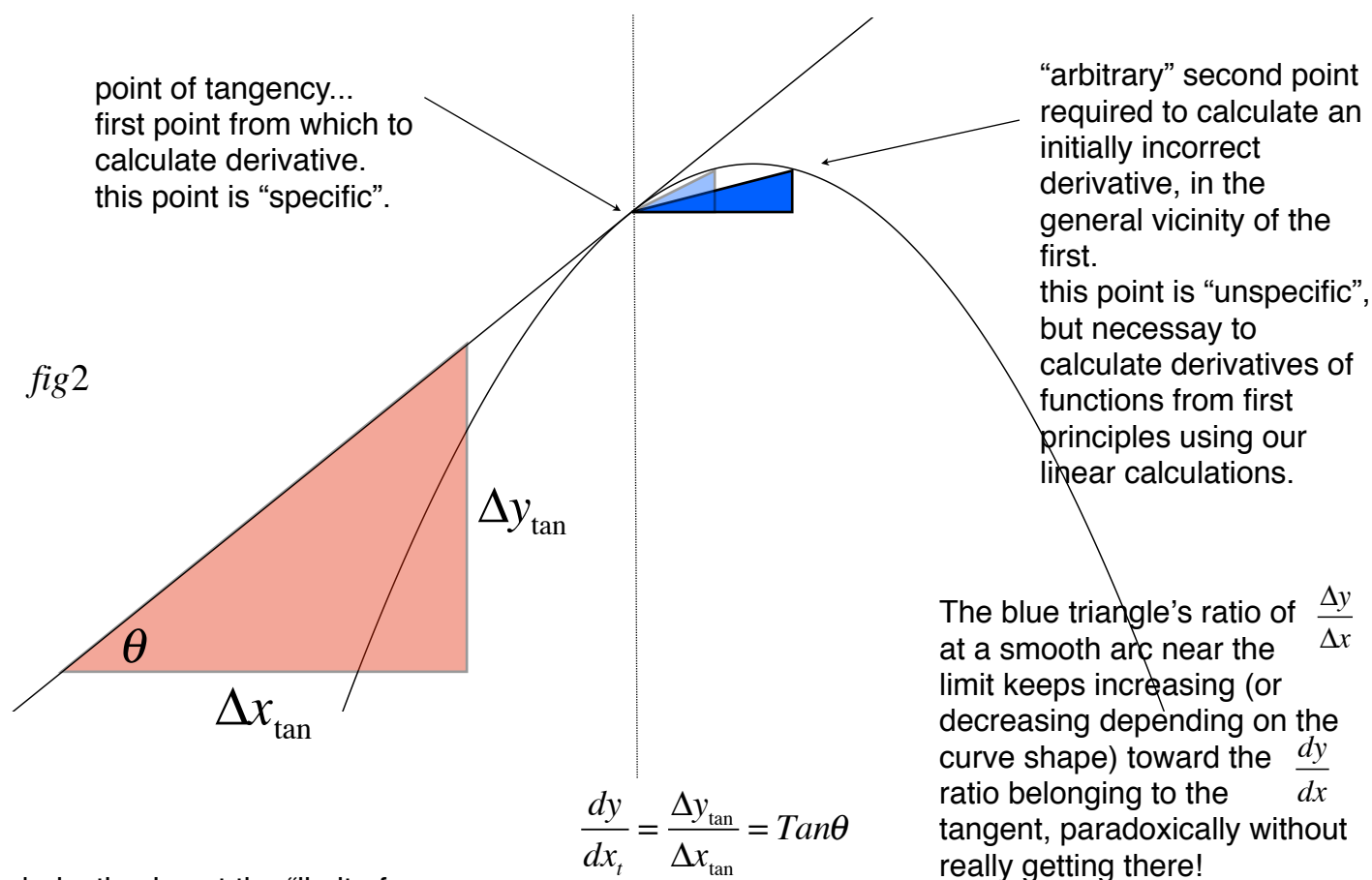A: Exactly, we start out being wrong but we will end up right. That second point will be on the curve. We will calculate the gradient of the line joining those 2 points, then we will hold the line fixed at the point of tangency, like the hands on a clock fixed in the middle of the clockface, then turn it until it completely overlaps the tangent itself.
B: This is like doing it by hand.
A: Yes, the math is built around that, expressing all of this in symbols, sometimes seeming contradictory and crazy, but we do it due to our dependence on lines, then finding a way to eliminate the error introduced by having to start off with a line that is not the tangent itself. Without being wrong in the beginning, we'd have to formulate a new way to write the math. We could of course do that but everyone is learning Calculus notation as formulated by the curious characters before us and at this stage we've no choice within this field.
Now it's time to give credit to the blue corner triangle's adversary, the red corner triangle.
B: I've had enough for a while. Time for a break.

point of tangency...
first point from which to
calculate derivative.
this point is "specific".

"arbitrary" second point
required to calculate an
initially incorrect
derivative, in the
general vicinity of the
first.
this point is "unspecific",
but necessay to
calculate derivatives of
functions from first
principles using our
linear calculations.

fig2

Δy_{tan}

θ

Δx_{tan}

The blue triangle's ratio of $\frac{\Delta y}{\Delta x}$ at a smooth arc near the limit keeps increasing (or decreasing depending on the curve shape) toward the $\frac{dy}{dx}$ ratio belonging to the tangent, paradoxically without really getting there!

$$\frac{dy}{dx_t} = \frac{\Delta y_{tan}}{\Delta x_{tan}} = Tan\theta$$

A derivative is not the "limit of a fraction", it is the ratio of the instantaneous rate of change of a function AT the limit itself, where the "limit" is where we wanted to be in the beginning and is only a limit due our linear approach to a non-linear question.
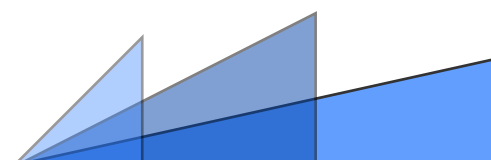
fig2 is a simple geometric representation of what happens at a limit in the case of a derivative for a single variable function. Hundreds of words can be used for students of varying levels and it's possible to teach this as soon as the concept of the difference between a linear and non-linear function is understood.

Having chosen the second point on the curve to write the initial "incorrect" gradient $\dfrac{f(x+\Delta x)-f(x)}{\Delta x}$ knowing that we should be dealing with a single point rather than 2, which is why the line gradient formula $\dfrac{y_2-y_1}{x_2-x_1}$ has been re-written in "first principles notation", since the 2nd point is being introduced deliberately for our linear notation, we proceed to eliminate the error from the "first principles" approximation.

Now, geometrically, we are performing a very simple operation. We begin with the blue line which enables use to write "a" gradient, but it's not the gradient we want, so we fix it to $f(x_t)$ and rotate it to overlap the red tangent.

fig3
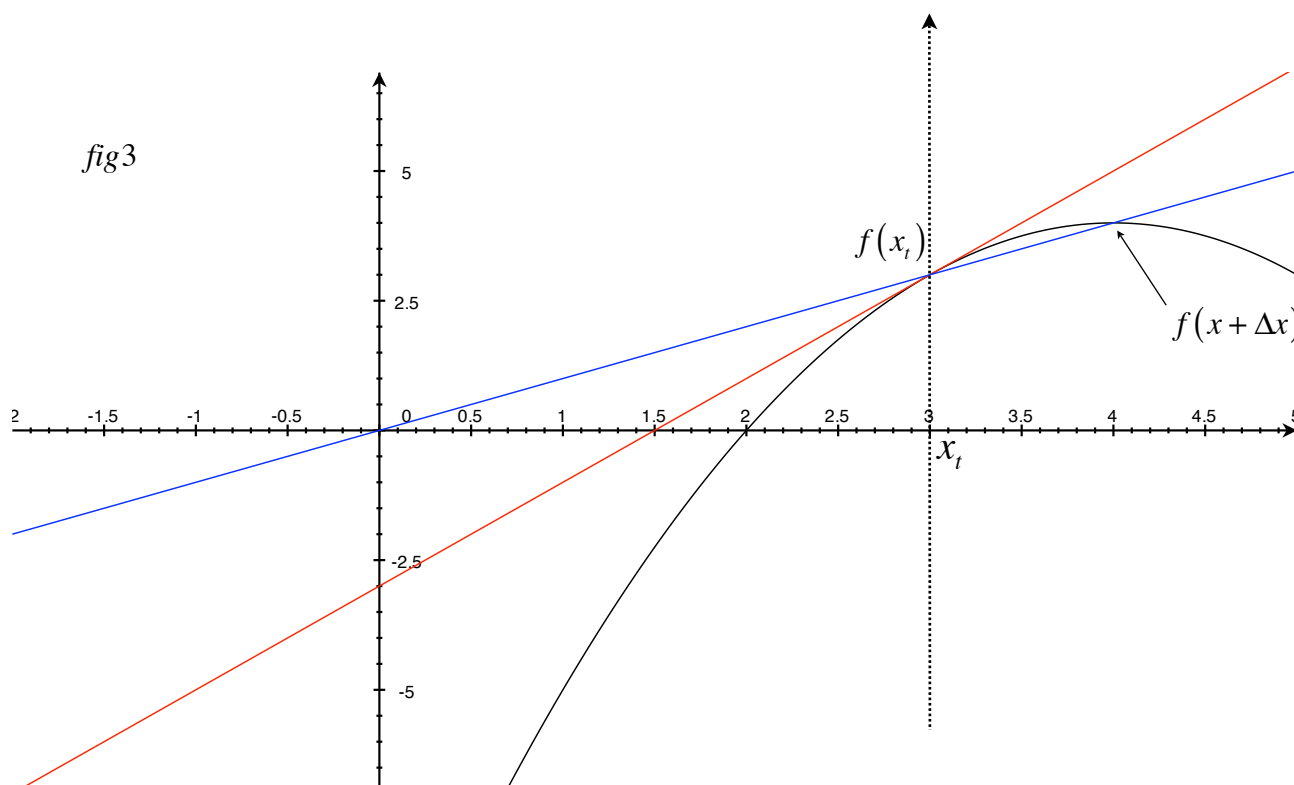
That's the geometry. We are truly interested only in the red line and the thing to appreciate is why we started with the blue innacurate one to end up at the red exact one.

The reason was to be able to "write" the mathematics in already familiar linear terms and from it develop the gradient of a line for which we have not 2 points but one.

From this perspective, it's not that easy to appreciate the mathematics of calculations for various levels of student, hence we bring in the red and blue right-angled triangles to accompany the red and blue lines. this gives a more comprehensive view of the mathematical formulation of simple geometry.

The reason we cannot simply rotate the blue line to the red one and write the gradient of the red line without "infinitesimally small values" is simply because the second point has to move along the curve until it collides with the first point (the point of tangency).

Mathematically it appears we are dealing with unmanageably small measurements, which occurs to the student who takes his eye off the geometry.
It does not matter how small the measurements of the "infinitesimals" become. We are concerned mainly with their ratio.

Refer back to fig2.
The blue triangle is "disappearing" as the second point on the curve approaches the point of tangency. It seems like an illusive triangle to deal with, but as it shrinks to zero size, when it reaches zero size, it is the smallest imagineable version of the red triangle.
At the limit, it's side ratios are the side ratios of the red triangle.
The red triangle is the blue triangle at $f(x_t)$ seen through a telescope!!
The blue triangle has shrunk to zero size but still exists.
We live in the present moment which is a "zero second time interval" yet the integral of these "zeroes" is a finite value. Hence, we face a situation of a NOT TRUE ZERO.
Enter Quantum Physics? No need. You see, we can imagine going to zero but for something that is continuous such as time or the real number system, is there ever zero? Something can keep getting smaller, without ever disappearing and though we may say, the blue triangle must disappear when the 2 points overlap, we are faced with the problem of the thickness of the point in which the blue triangle exists.
The blue triangle disappears if the point of tangency disappears. If that disappears, the line is no longer a tangent to the circle as there is no longer a point of tangency, since if every point has zero thickness we can not view any geometry at all!

We have the same situation in the circle, where we calculate circle area with right-angled triangles whose size approaches zero. We calculate π with a similar technique. We start with a square and turn into into a circle. We can say π is the number of diameters in a circumference of a circle, or the ratio of circle circumference to the side of the square that it is the "incircle of", or the area of a circle to a quarter of the area of the square it is an incircle of, another linear to non-linear ratio. It is no surprise that π is a real value impossible to pin down and it's "linear" formula is

$$\lim_{N \to \infty} NSin\left(\frac{180°}{N}\right) \leq \pi \leq \lim_{N \to \infty} N\tan\left(\frac{180°}{N}\right)$$

So the blue triangle at the limit magnified greatly is the red triangle.

Sorry about the anticlimax, the blue triangle lives on!
Still, the objective was to try to hold the attention of the inquisitive student to the end.

Now, we still have to examine the mathematics of the derivative.
Let's proceed step by step...

$\dfrac{y_2 - y_1}{x_2 - x_1}$ or $\dfrac{y - y_1}{x - x_1}$ or $\dfrac{f(x) - f(x_1)}{x - x_1}$ etc is the gradient of a line using 2 points on it.

$\dfrac{f(x + \Delta x) - f(x)}{\Delta x}$ makes the 2nd point "loose" and is the beginning of writing the tangent

gradient using only the point of tangency! It is the ratio of the $\perp$ sides of the blue triangle.

$\lim_{\Delta x \to 0} \dfrac{f(x + \Delta x) - f(x)}{\Delta x}$ is that ratio after we eliminate the error introduced by purposely

choosing the wrong line from which to derive the tangent gradient initially. It is the ratio of

the sides of the red triangle IF WE CAN MATHEMATICALLY DERIVE IT!!

The red triangle may be increased or reduced to any size we like to view it at.

It will always be the exact same shape.

Mathematically this is expressed as permanent $Sin\theta$, $Cos\theta$ and $Tan\theta$ ratios.

$$Tan\theta \text{ is } \frac{dy}{dx}$$

The blue triangle arises through the notation of our linear conscious thinking as we have begun to formulate the instantaneous rate of change of our non-linear function. This triangle has variable side proportions and never gives us $\frac{dy}{dx}$ until it disappears entirely!!! This triangle is necessary to write the mathematics in symbols and it's "goal" is to be resurrected as the red triangle.

We need to weigh the merits of both to appreciate the wonderful development of calculus and understand unambiguously it's resolution of the controversy of calculus notation.

$\frac{dy}{dx}$ is the ratio of dy to dx not just fancy notation. dy is the length of the vertical side of the blue triangle and dx is the length of the horizontal side of the blue triangle "at the point of tangency".

As we just discovered, it never actually disappears, hence dy and dx are measureable but more specifically, they form a specific ratio or fraction.

They are as small as small can be but they form a ratio.

Can you play around with them like regular fractions?

Of course! If you are at a "particular point" on the curve. Don't do it where the gradients differ! The ratios will not be the same and to be truthful, you should be modifying your notation to indicate that, such as dx_{t1} and dx_{t2} instead of dx generally.

After all, we write $\frac{dy}{dx} = f'(x)$ so $dy = f'(x)dx$ so $\int dy = \int f'(x)dx = y$

Admittedly, this depicts a "leaning tower of Pisa totem pole". Now it's your turn to draw that diagram.

There is an episode of the original Star Trek series, in which Kirk navigated the Enterprise to the "Edge of the known Universe" to converse with his creator. This viewpoint defines "Universe" as being a bounded region enclosing the enclosed and the creator of it as being outside of it possibly. According to the "Big Bang" theory, the Universe we live in, if we live in one, depending on how we think of the meaning of "Universe", should be finite and expanding. This is a very limited viewpoint. We then have issues such as ...what initiated the big bang, apart from saying "it should have happened as the stars appear to be receding", did life even have a beginning or is it a form of continuum with no beginning and no end and so on and on and on.
If we realise how we think, we see the limitations of our viewpoints.
We have the same type of dichotomy at the derivative of a function in calculus.
Here's what I mean by that.

Imagine boarding your spaceship and your power source is a perfect anti-matter generator, self regulating, self maintaining, a flawless design, as you live in an age where the perfect theory spawns the perfect design.
You travel from your point of departure and voyage to infinity.
You will never get there of course since once you "reach it", you realise you are deluding yourself and infinity is "as far away from you now as when you started your voyage". You return home feeling embarrassed.
On your journey out, there came a point where nobody could see you, you were out of view, but your size never changed in your eco-cockpit, the impression of your size continually reducing was caused by your increasing distance from your origin.
This is the exact same situation we have with the red triangle in fig1 and fig2.
It is indestuctible as we can continually reduce it in a never-ending continuum. It "disappears" from "our view" without ever reaching true zero at the point of tangency. Geometrically we can visualise this in a fraction of a second but contention can arise in the mathematical formulation, as we do this using the blue triangle, which changes shape as it reduces in size, though it's really only changing shape "imperceptibly" at the microscopic region of the limit. Imagine the triangles moving away from you in 3-D.
The trouble is, the red and blue triangles do truly meet at infinity!!! but they can't! logically. from this limited viewpoint!! This is because the blue triangle does continue to change shape as it reduces in size!! albeit "imperceptibly" but it's very clear imaginatively.

The mathematics of derivatives, using first principles is symbolised "in terms of the blue triangle".
The exact value of the derivative is given by the red triangle. If the trianges meet at infinity, they don't meet at all. Yet, we said earlier they do!!
Of course they do, but not if we are approaching a limit since we are reducing indefinately.

Look back again at fig3. If we say "we move the blue line pivoted at the point of tangency until it overlaps the red tangent, then the derivative of the function at that point is the gradient of the tangent". No reference to limits.
But the mathematics of calculus formulates by starting with the gradient of the blue line using points on the curve. We don't have to focus in on the values going to zero, but we do have to "deal with them" by cancelling them from numerator and denominator, which we could never do if we weren't dealing with fractions! or other means. It's all a symptom of how the math was developed and our initial linear point of view.

Instead of $\quad \dfrac{dy}{dx} = \lim_{\Delta x \to 0} \dfrac{f(x + \Delta x) - f(x)}{\Delta x}$

we could write $\quad \dfrac{dy}{dx} = \; gradient \; of \; tangent \; to \; f(x) \; at \; y = f(x_t)$

If we were inclined, we could now spend our time geometrically calculating derivatives for various functions but could we choose every point of tangency??
That question answers itself, whereas the function formula is constant over the domain of the variable, so that's the power of the math behind calculating derivatives in the non-obvious way and there is no remaining conflict once it has been resolved by knowing the limit can always be seen clearly on the tangent, if we want to look at an example or two. Imaginatively, you say to yourself, "it doesnt matter what gradient the tangent is, it gives the derivative and it's very clear what is happening for a vertical tangent, we have a discontinuous function in that case".
We can say there is a "discontinuity" at zero between the mathematics of first principles and the simple geometric truth, due to the behaviour of the blue triangle.
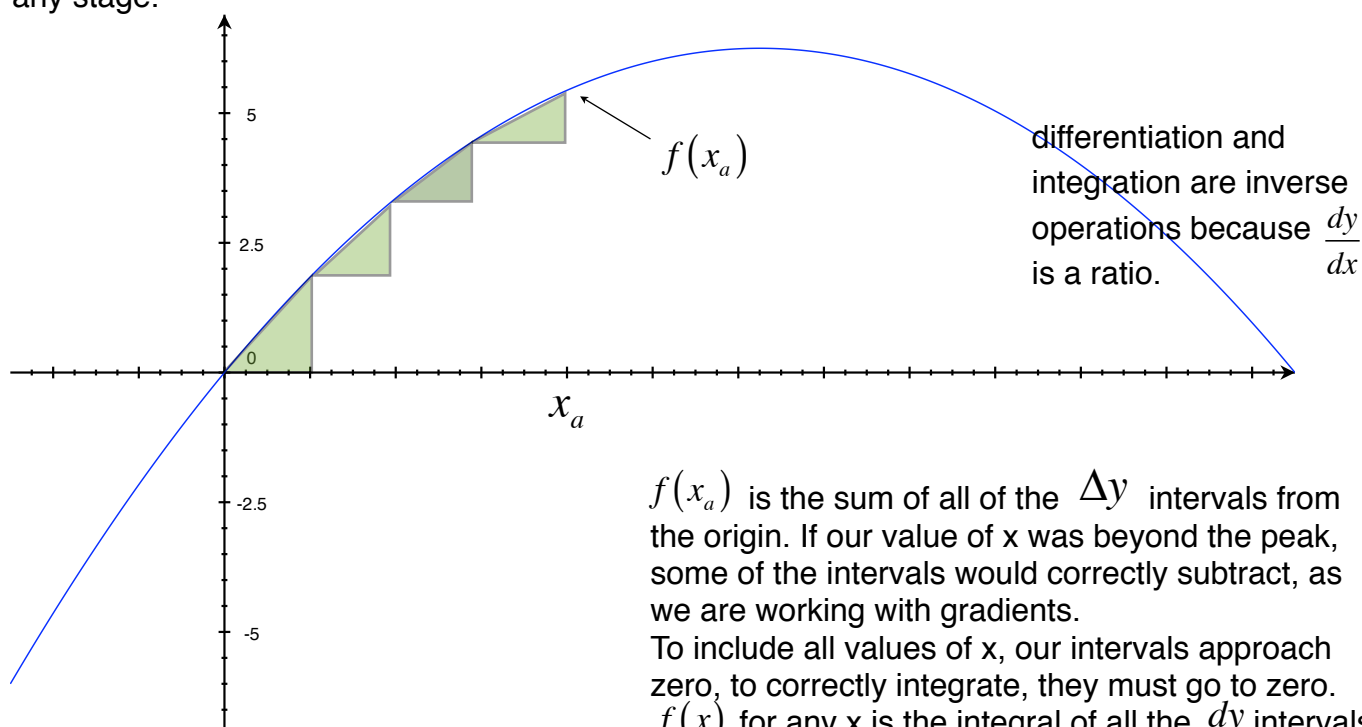Reconciling this is very easy when you see the whole picture of where a student may try working only with the math, without keeping the geometry of the lines in fig3 in mental view. He will certainly have questions to resolve if he takes a one-pointed approach to what the mathematics is attempting to achieve and very successfully achieve.

Remember how Nikola Tesla developed alternating current to improve resoundingly on "obvious DC"? He took a non-obvious approach and designed his entire scheme mentally without mathematics formulae. Once designed it could be formulated and was, but he was the one with the insight. He originated, others followed and used words and diagrams to explain.

Derivatives are a matter of "mathematically legally" removing the tiny increments of Δy and Δx that we needed to introduce to be able to write the tangent gradient.
We had to introduce them, now we must get rid of them!!

If a student tries to be insightful about it, the subject notation will cause no confusion at any stage.

$f(x_a)$

x_a

differentiation and integration are inverse operations because $\frac{dy}{dx}$ is a ratio.

$f(x_a)$ is the sum of all of the Δy intervals from the origin. If our value of x was beyond the peak, some of the intervals would correctly subtract, as we are working with gradients.
To include all values of x, our intervals approach zero, to correctly integrate, they must go to zero. $f(x)$ for any x is the integral of all the dy intervals up to that x. dy is Δy that forms a ratio with Δx as $\Delta x \to 0$ such that the ratio at the limit equals the gradient of the tangent.
So, $\int dy = f(x)$ but $f'(x) = \frac{dy}{dx}$ at x

so $f'(x)dx = dy$ and $\int f'(x)dx = \int dy = y$ or $f(x)$

It's only through understanding what's happening geometrically that we can clearly understand that differentiation and integration are not "accidentally" inverse operations but because $\frac{dy}{dx}$ is a ratio, though varying.

IN TERMS OF THE DERIVATIVE, A "SINGLE POINT OF TANGENCY" IN THE
CONTINUUM OF THE REAL NUMBER SYSTEM MASKS THE GEOMETRIC REALITY

Referring back to fig2, if we "imagine" the real number system to be continous, we can
imagine the blue triangle reducing, reducing, reducing, never reaching zero and so never
really emerging with the red triangle. This does however require that we simultaneously
imagine the 2nd point on the blue line approaching the point of tangency at a reducing
speed!

Do you follow?
Yes, it's got to slow down dramatically so that it's speed approaches zero.
Does this not seem completely impractical?
It's a symptom of the real number system being a continuum.
Now consider this. We cannot isolate the "smallest possible size". We can't imagine it. We
can wonder about it but whatever scale we visualise it at, it can still be 100 times smaller
again and again and again "in our imagination".
<u>There is a flaw in this analysis</u>, which we haven't pointed out yet!
We have concentrated on examining the limit at zero, but in Calculus we never need go
there due to the fact that we are dealing with a tangent!!
We must, in approaching zero, examine the change in curvature of the curve.

$fig4$

$f(x + \Delta x)$

Within a "tangential" frame of
measurement, the blue triangle
unambiguously merges with the red
one at the "point" of tangency, or the
"infinitesimal region of tangency".
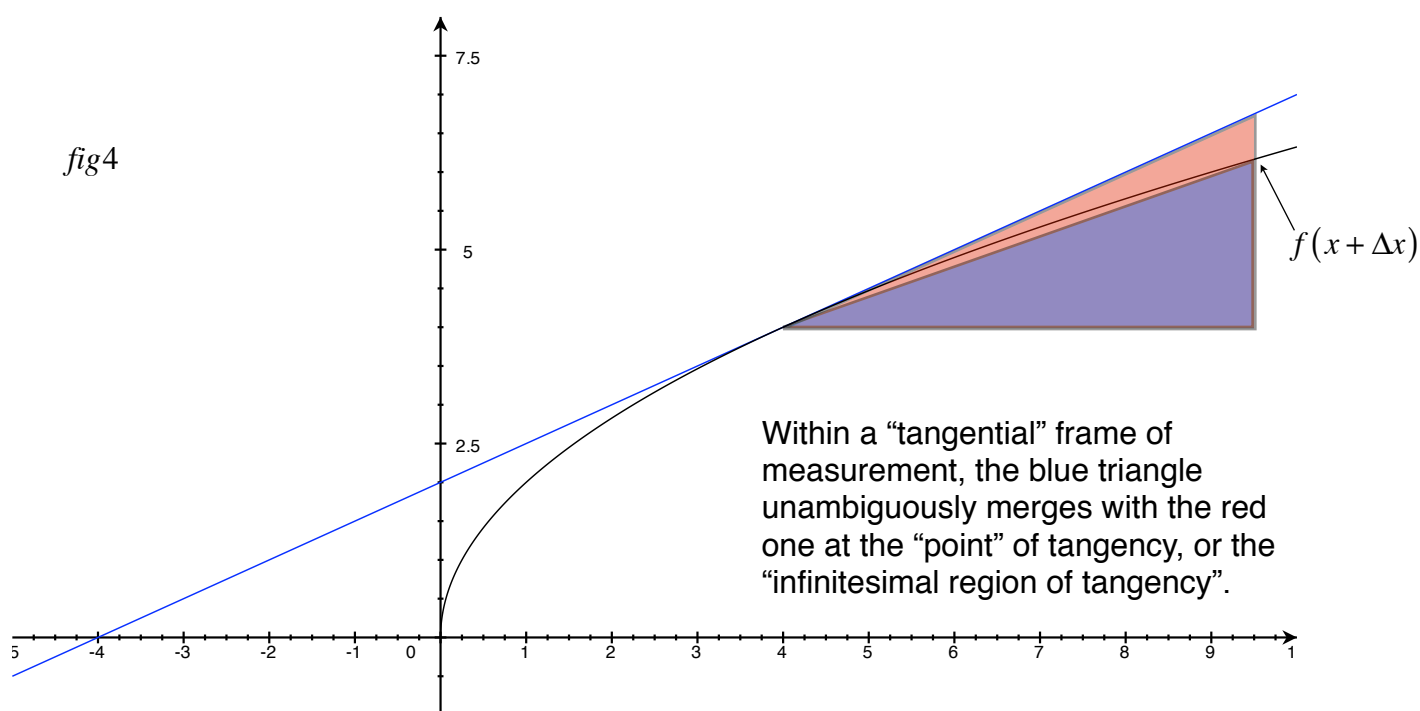
We have been analysing with lines and triangles.
Now for the "coup de grace"...... bring on the CIRCLE.

Any part of a curve may be viewed at certain depth as an ARC OF A CIRCLE.
First visualise this statement and determine whether or not you can find an exception that
is not a discontinuity of curvature, such as an edge which is straight.

This being so, we can now view the following circle, centred at (3,0) with a radius of 3, at
the origin for various levels of magnification.

It intersects the y-axis at y=0, a single unique point of tangency.

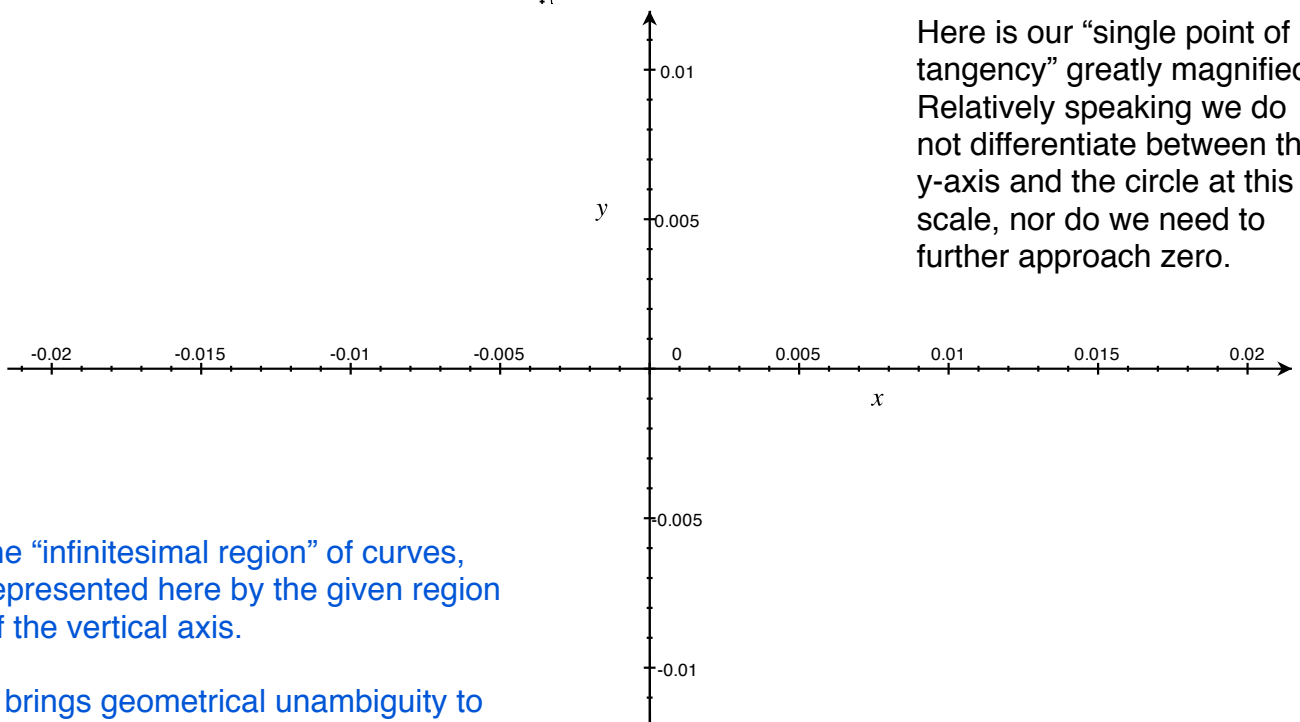$$(x-3)^2 + y^2 = 9$$

At zero, the circle's direction is north-south.
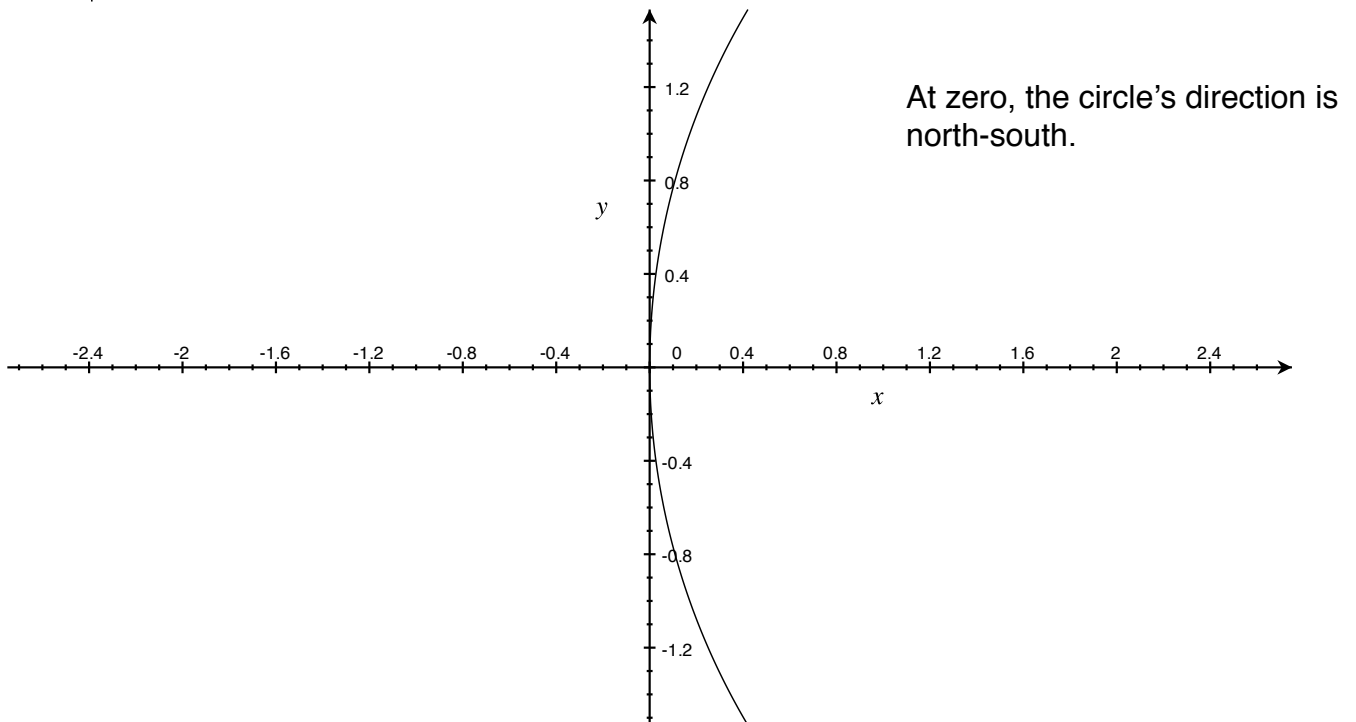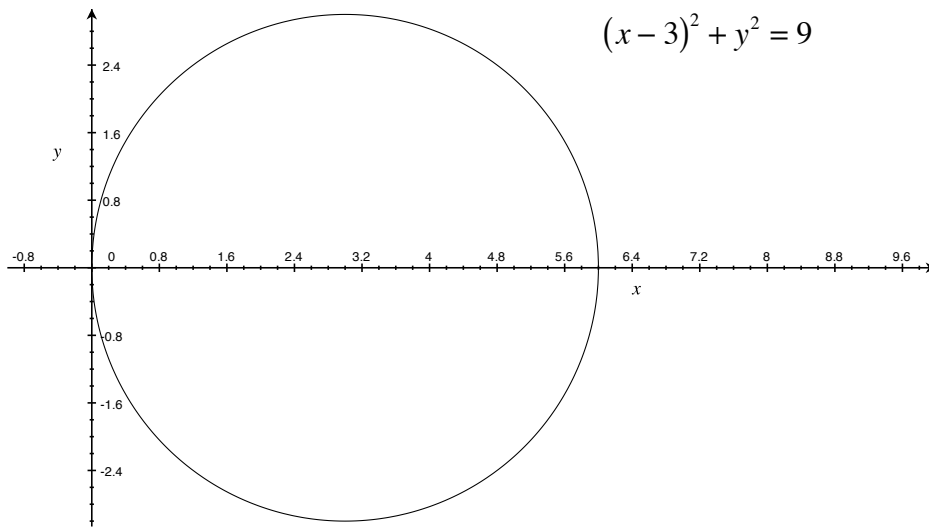
As we "zoom in", we cannot avoid the circle overlap the y-axis at the "single point of tangency".
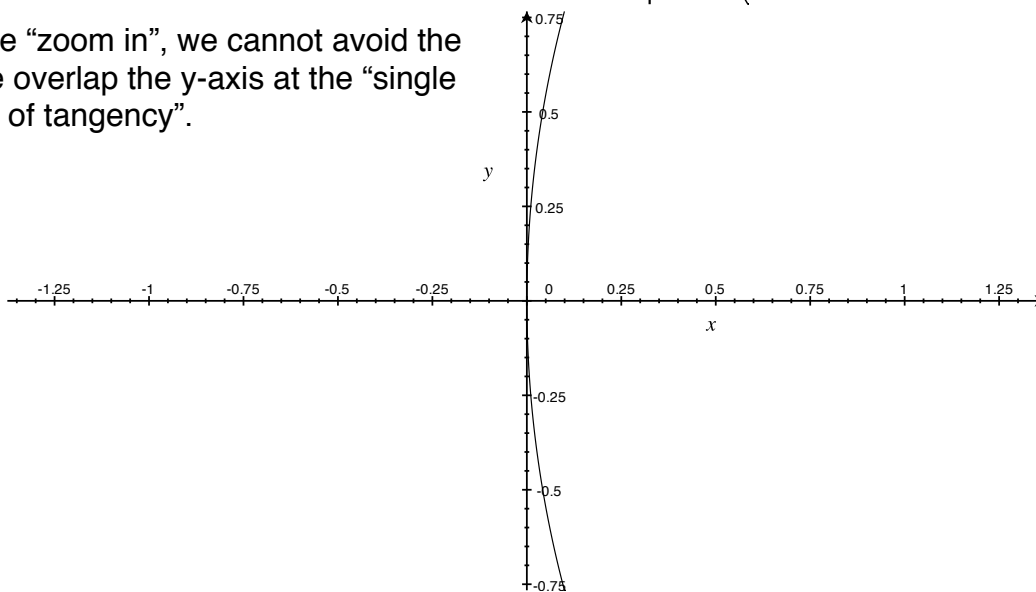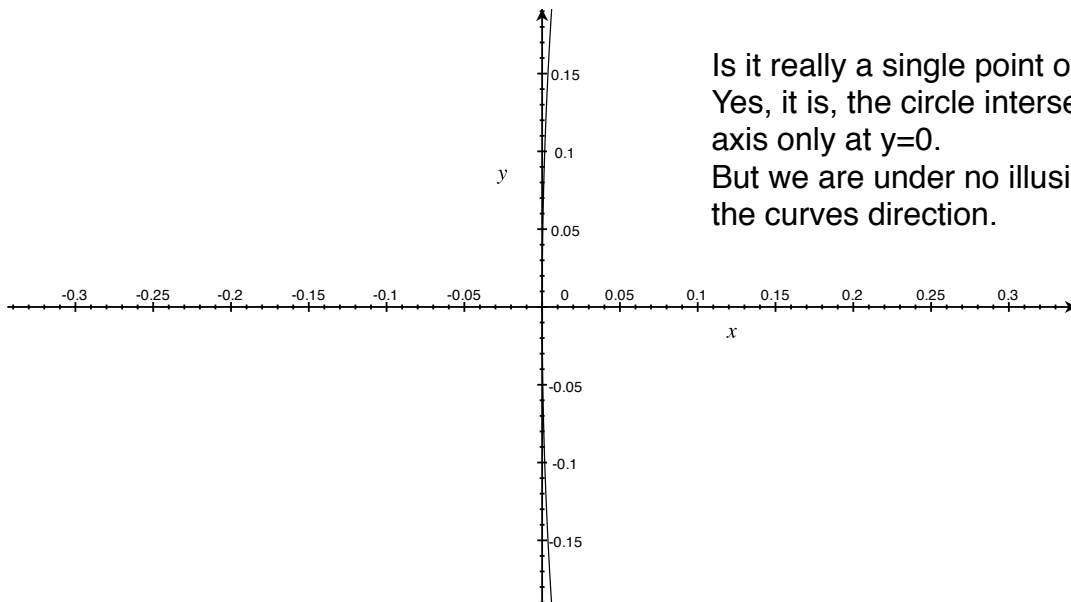
Is it really a single point of tangency?
Yes, it is, the circle intersects the y-axis only at y=0.
But we are under no illusion about the curves direction.

Here is our "single point of tangency" greatly magnified. Relatively speaking we do not differentiate between the y-axis and the circle at this scale, nor do we need to further approach zero.

the "infinitesimal region" of curves, represented here by the given region of the vertical axis.

It brings geometrical unambiguity to the derivative.

We may stop our analysis at the level of the "infinitesimal" since we are dealing with a tangent. The point of intersection of perpendicular lines on the other hand occurs at "zero-width" location. Derivatives are unconcerned with that.
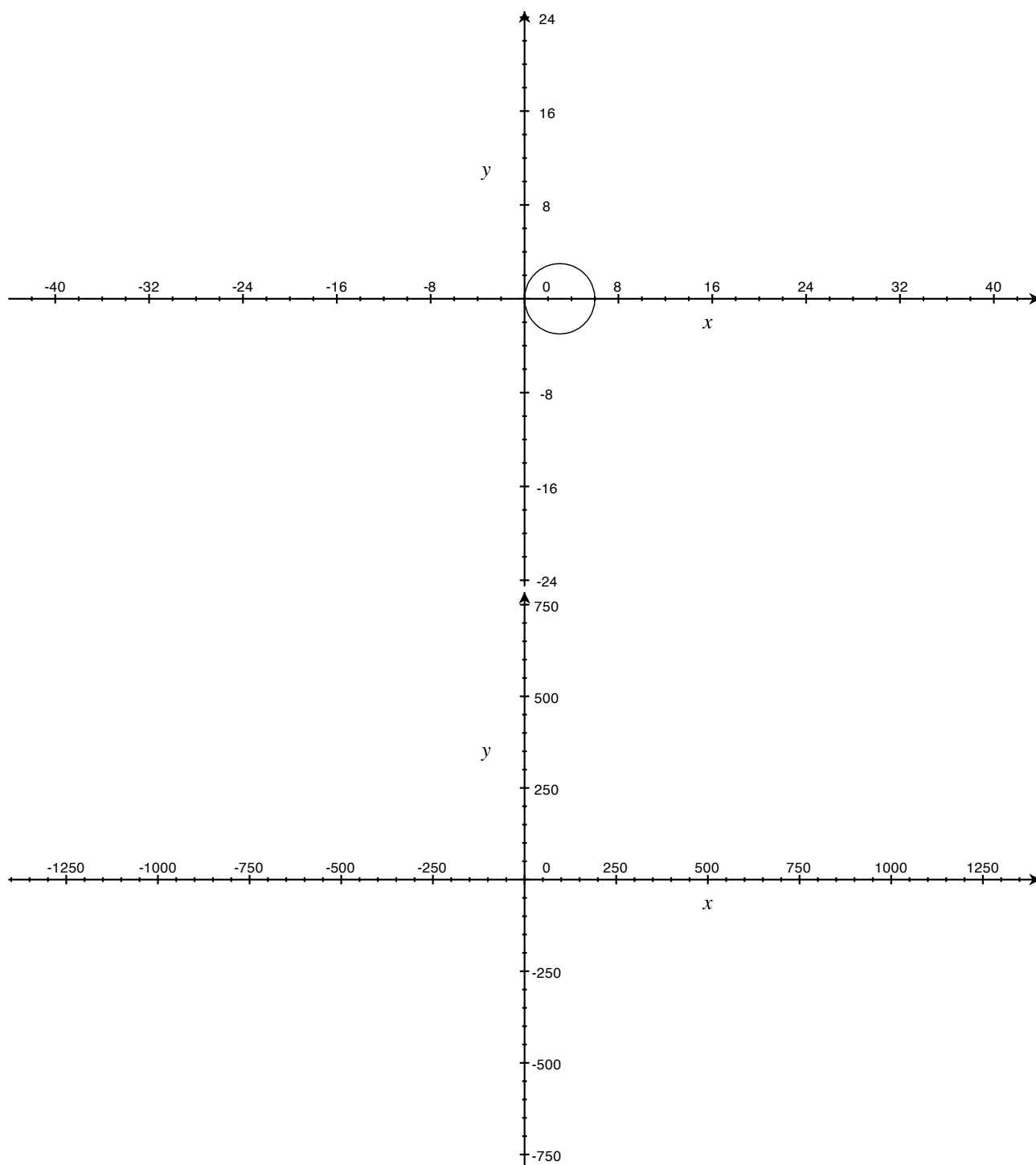
At another extreme of scale, we cannot geometrically examine the curve. This is why the real number system is necessary. The circle is there, we just can't see it, nor examine it. Nothing can be "determined". This shows that "any shape" exists at "any point"!