Suppose M, N, K are left R-modules and

$$M \overset{\varphi}{\rightarrow} N \overset{\psi}{\rightarrow} K \rightarrow 0$$

is an exact sequence of R-homeomorphisms, Then for any right R-module T the following sequence

$$T \otimes M \xrightarrow{1_T \otimes \varphi} T \otimes N \xrightarrow{1_T \otimes \psi} T \otimes K \rightarrow 0$$

is an exact sequence of $\mathbb{Z}$-homeomorphisms. Moreover, if R is commutative, then it becomes an exact sequence of R-homeomorphisms.

Proof:

Well, I omit writing the proof completely. I just write down the parts that I can't understand. First it proves that $Im(1_T \otimes \psi) = T \otimes K$ and then it shows that $Im(1_T \otimes \varphi) \subseteq Ker(1_T \otimes \psi)$. Now it wants to show that $Im(1_T \otimes \varphi) \supseteq Ker(1_T \otimes \psi)$ to conclude that the given sequence is exact.

Well, first it defines $\pi: T \otimes N \rightarrow \frac{T \otimes N}{Im(1_T \otimes \varphi)}$ by $\pi(x \otimes z) = \overline{x \otimes z}$ where $x \in T \ and \ z \in N$, then it defines:

$f: T \times K \rightarrow \frac{T \otimes N}{Im(1_T \otimes \varphi)}$ by $f(x, y) = \pi(x \otimes z) : y = \psi(z)$ and it shows that f is well-defined.

It could be verified easily that f is linear in each argument, so there exists a unique $\mathbb{Z}$-homeomorphism $\varphi_1: T \otimes K \rightarrow \frac{T \otimes N}{Im(1_T \otimes \varphi)}$ such that the following diagram

$$\begin{tikzcd}
T \times K \arrow[r, "\otimes"] \arrow[d, "f"'] & T \otimes K \arrow[dl, dashed, "\varphi_1"] \\
\dfrac{T \otimes N}{Im(1_T \otimes \varphi)} &
\end{tikzcd}$$

commutes. Hence, $\varphi_1 \otimes = f$ and we conclude that:

$$\varphi_1(x \otimes y) = \pi(x \otimes z) \; where \; y = \psi(z) and \; z \in N.$$

On the other hand, if we define $\varphi_2 \colon \frac{T \otimes N}{Im(1_T \otimes \varphi)} \to T \otimes K$ by:

$$\varphi_2\left(\pi\left(\sum_{i=1}^{t} x_i \otimes y_i\right)\right) = \sum_{i=1}^{t} x_i \otimes \psi(y_i)$$

Then $Im(1_T \otimes \varphi) \subseteq Ker(1_T \otimes \psi)$ shows that φ_2 is well defined (This is where I don't understand) and therefore φ_2 turns into a $\mathbb{Z}$-homeomorphism. Now, It's easy to verify that $\varphi_1\varphi_2$ and $\varphi_2\varphi_1$are both identity; hence φ_2 is a $\mathbb{Z}$-isomorphism. (I checked this part and I understand this one) and therefore it tells us that:

$$Ker(1_T \otimes \psi) \subseteq Im(1_T \otimes \varphi)$$ (I don't understand this one too)

The rest of the proof is easy to understand. I just don't understand the parts that I highlighted in red.

Thanks in advance.