I’ve defined a new map called f as the following:

$$f:\frac{R}{I}\times M \to \frac{M}{Im\big(\varphi(j\otimes 1_M)\big)}\ ,f(\bar{r},x)=\pi'(rx)$$

Where π' is the natural epimorphism $\pi': M \to \frac{M}{Im(\varphi(j\otimes 1_M))}$ that sends rx to its coset. (note that it’s different than $\pi: R \to R/I$ that is used in the proof by the book).

f is well-defined, because:

$$\overline{r_1}=\overline{r_2}\to\overline{r_1}-\overline{r_2}=0\to\overline{r_1-r_2}=0$$

$$\Rightarrow f(\bar{0},x)=f(\overline{r_1-r_2},x)=\ \pi'(\,(r_1-r_2)x)=\pi'(r_1x-r_2x)=\pi'(r_1x)-\pi'(r_2x)=f(\bar{0},x)=\pi'(0x)=\pi(0)=0$$
$$\to\ \ \pi'(r_1x)=\pi'(r_2x)\to f(\overline{r_1},x)=f(\overline{r_2},x)$$

Therefore f is well-defined.

f is a bilinear function as well because:

$$f(\overline{r_1}+\overline{r_2},x)=f(\overline{r_1+r_2},x)=\pi'(\,(r_1+r_2)x)=\pi'(r_1x+r_2x)=\pi'(r_1x)+\pi'(r_2x)=f(\overline{r_1},x)+f(\overline{r_2},x)$$

$$f(\bar{r},x+x')=\pi'\big(r(x+x')\big)=\pi'(rx+rx')=\pi'(rx)+\pi'(rx')=f(\bar{r},x)+f(\bar{r},x')$$

$$f(\bar{r}.s,x)=f(\overline{r.s},x)=\pi'\big((rs).x\big)=\pi'\big(r.(sx)\big)=f(\bar{r},sx)$$

Note that $\bar{r}.s=\overline{r.s}$ is a well-defined scalar product because $r\in I$ and I is a right ideal of R.

Now, according to a theorem proved earlier, (or actually the definition of tensor product in some books), a unique homomorphism map φ is induced such that the following diagram commutes:

$$\begin{tikzcd}
\frac{R}{I}\times M \arrow[r, "\otimes"] \arrow[d, "f"'] & \frac{R}{I}\otimes M \arrow[dl, dashed, "\varphi_1"] \\
\frac{M}{Im\big(\varphi(j\otimes 1_M)\big)} &
\end{tikzcd}$$

This tells us that $\varphi_1\otimes=f$ or $\varphi_1\otimes(\bar{r},x)=\varphi_1(\bar{r}\otimes x)=\pi'(rx)$. Now let’s define:

$\varphi_2: \frac{M}{Im(\varphi(j\otimes 1_M))}\to\frac{R}{I}\otimes M$ by $\varphi_2\big(\pi'(rx)\big)=\bar{r}\otimes x$. Note that it’s enough to prove that this function is well-defined, because if so, then $\varphi_1\varphi_2=\varphi_2\varphi_1=identity$ which tells us that φ_1 is bijective and hence $\frac{R}{I}\otimes M\cong\frac{M}{Im(\varphi(j\otimes 1_M))}$

To prove that φ_2 is well-defined, assume $rx=r'x'$:

$$\pi'(rx)=\pi'(r'x')\to\pi'(rx)-\pi'(r'x')=0\to\ \pi'(rx-r'x')=0\to rx-r'x'\in\ker\pi'$$

But according to the definition of π', $\ker\pi'=Im\big(\varphi(j\otimes 1_M)\big)\subseteq\ker(\,(\pi\otimes 1_M)\varphi^{-1})$

I've shown somewhere else that $Im(\varphi(j \otimes 1_M)) \subseteq \ker((\pi \otimes 1_M)\varphi^{-1})$ but here I repeat it for convenience:

$$\forall x \in Im(\varphi(j \otimes 1_M)), \exists iy \in IM: x = i.y \rightarrow \ (\pi \otimes 1_M)\varphi^{-1}(i.y) = (\pi \otimes 1_M)(i \otimes y) = \pi(i) \otimes y = \ \bar{0} \otimes y = 0$$

Where $i \in I, y \in M$. (Note that the book itself has shown that $Im(\varphi(j \otimes 1_M)) = IM$).

OK. Now, let's return to the proof that φ_2 is well-defined,

$$\pi'(rx) = \pi'(r'x') \rightarrow rx - r'x' \in \ker \pi' \rightarrow rx - r'x' \in \ker((\pi \otimes 1_M)\varphi^{-1})$$

$$(\pi \otimes 1_M)\varphi^{-1}(rx - r'x') = (\pi \otimes 1_M)(\varphi^{-1}(rx) - \varphi^{-1}(r'x')) = (\pi \otimes 1_M)(r \otimes x - r' \otimes x') = \bar{r} \otimes x - \overline{r'} \otimes x = 0$$
$$\rightarrow \ \bar{r} \otimes x = \ \overline{r'} \otimes x \rightarrow \varphi_2(\pi'(rx)) = \varphi_2(\pi'(r'x'))$$

OK. φ_2 is well-defined. And it's now proved that $\frac{R}{I} \otimes M \cong \frac{M}{Im(\varphi(j \otimes 1_M))}$, but $Im(\varphi(j \otimes 1_M)) = IM$, finally:

$$\frac{R}{I} \otimes M \cong M/IM$$

Q.E.D