

Theorem: Suppose that  $R$  is a ring and  $I$  is a right ideal of  $R$ . if  $M$  is a left  $R$ -module, then:

$$\frac{R}{I} \otimes M \cong \frac{M}{IM} \text{ (}\mathbb{Z}\text{-isomorphic)}$$

Moreover, If  $R$  is commutative, they'll be  $R$ -isomorphic.

Proof: Consider the exact sequence  $I \xrightarrow{j} R \xrightarrow{\pi} R/I \rightarrow o$ . By a theorem that we proved earlier the following sequence

$$I \otimes M \xrightarrow{j \otimes 1_M} R \otimes M \xrightarrow{\pi \otimes 1_M} \frac{R}{I} \otimes M \rightarrow o$$

Is an exact sequence of  $\mathbb{Z}$ -homomorphisms. But according to the theorem we just proved, there exists a  $R$ -homeomorphism  $\varphi: R \otimes M \rightarrow M$ ,  $\varphi(r \otimes x) = rx$  that guarantees  $R \otimes M \cong M$ . Now we could easily show that the sequence:

$$I \otimes M \xrightarrow{\varphi(j \otimes 1_M)} M \xrightarrow{(\pi \otimes 1_M)\varphi^{-1}} \frac{R}{I} \otimes M \rightarrow o$$

is an exact sequence of  $\mathbb{Z}$ -homomorphisms.

Now according to the first isomorphism theorem for modules:

$$\frac{R}{I} \otimes M \cong \frac{M}{\text{Ker}((\pi \otimes 1_M)\varphi^{-1})} \cong \frac{M}{\text{Im}(\varphi(j \otimes 1_M))}$$

$$\begin{aligned} \text{Im}(\varphi(j \otimes 1_M)) &= \left\{ \varphi(j \otimes 1_M) \left( \sum_{i=1}^t x_i \otimes y_i \right) : x_i \in I, y_i \in M \right\} \\ &= \left\{ \varphi \left( \sum_{i=1}^t x_i \otimes y_i \right) : x_i \in I, y_i \in M \right\} = \left\{ \sum_{i=1}^t x_i y_i : x_i \in I, y_i \in M \right\} = IM \end{aligned}$$

This proves that  $\frac{R}{I} \otimes M \cong M/IM$ . Moreover, If  $R$  is commutative, it could be easily shown that all the homomorphism we used are  $R$ -homomorphisms. Q.E.D

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So it has left the part that I've highlighted in red to the reader. I'm trying to show that that sequence is exact.

Since  $\varphi$  is bijective (It was an isomorphism), we know that  $\varphi^{-1}$  is surjective.  $\pi \otimes 1_M$  is surjective as well and from set theory we know that the composition of surjective functions is a third. Therefore  $(\pi \otimes 1_M)\varphi^{-1}$  is surjective.

Now I must prove that  $\text{Im}(\varphi(j \otimes 1_M)) = \text{Ker}((\pi \otimes 1_M)\varphi^{-1})$

It's enough that I prove  $\text{Ker}((\pi \otimes 1_M)\varphi^{-1}) = IM$ . But  $RM=M$  Therefore:

$$\forall x \in M, \exists ry \in RM: x = ry \rightarrow (\pi \otimes 1_M)\varphi^{-1}(x) = (\pi \otimes 1_M)\varphi^{-1}(ry) = (\pi \otimes 1_M)(r \otimes y) = (r + I) \otimes y$$

Now if this thing wants to be zero,  $r + I$  or  $y$  must be zero. If  $y$  is zero then  $x$  is zero, if  $r + I$  is zero then it must be  $r + I = I$  which means  $r \in I$ .

Therefore,  $\text{Ker}((\pi \otimes 1_M)\varphi^{-1}) = IM$  and the sequence is exact.