The problem:

$y''' + 8y = 2x - 5 + 8e^{-2x};\ y(0) = -5,\ y'(0) = 3, y''(0) = -4$

My work:

$m^3 + 8 = 0 \Rightarrow m = -2 \Rightarrow (m^3 + 8) \div (m + 2) = m^2 - 2m + 4 \Rightarrow (m + 2)(m^2 - 2m + 4) = 0$

$\Rightarrow m = -2, \dfrac{2 \pm \sqrt{(-2)^2 - 4(1)(4)}}{2(1)} \Rightarrow m = -2,\ 1 \pm \sqrt{3}i \Rightarrow y_c = c_1 e^{-2x} + e^x\left(c_2 \cos(\sqrt{3}x) + c_3 \sin(\sqrt{3}x)\right)$

$y_c' = -2c_1 e^{-2x} + e^x\left(c_2 \cos(\sqrt{3}x) + c_3 \sin(\sqrt{3}x)\right) + e^x\left(-\sqrt{3}c_2 \sin(\sqrt{3}x) + \sqrt{3}c_3 \cos(\sqrt{3}x)\right)$

$y_c'' = 4c_1 e^{-2x} + e^x\left(c_2 \cos(\sqrt{3}x) + c_3 \sin(\sqrt{3}x)\right)' + e^x\left(c_2 \cos(\sqrt{3}x) + c_3 \sin(\sqrt{3}x)\right) + e^x\left(-\sqrt{3}c_2 \sin(\sqrt{3}x) + \sqrt{3}c_3 \cos(\sqrt{3}x)\right)' + e^x\left(-\sqrt{3}c_2 \sin(\sqrt{3}x) + \sqrt{3}c_3 \cos(\sqrt{3}x)\right)$

$y_c'' = 4c_1 e^{-2x} + e^x\left(-\sqrt{3}c_2 \sin(\sqrt{3}x) + \sqrt{3}c_3 \cos(\sqrt{3}x)\right) + e^x\left(c_2 \cos(\sqrt{3}x) + c_3 \sin(\sqrt{3}x)\right) + e^x\left(-3c_2 \cos(\sqrt{3}x) - 3c_3 \sin(\sqrt{3}x)\right) + e^x\left(-\sqrt{3}c_2 \sin(\sqrt{3}x) + \sqrt{3}c_3 \cos(\sqrt{3}x)\right)$

$y_c'' = 4c_1 e^{-2x} + 2e^x\left(-\sqrt{3}c_2 \sin(\sqrt{3}x) + \sqrt{3}c_3 \cos(\sqrt{3}x)\right) + e^x\left(c_2 \cos(\sqrt{3}x) + c_3 \sin(\sqrt{3}x)\right) + e^x\left(-3c_2 \cos(\sqrt{3}x) - 3c_3 \sin(\sqrt{3}x)\right)$

$y_p = Ax + B + \left(Cxe^{-2x}\right)$

$y_p' = A + \left(Ce^{-2x} - 2Cxe^{-2x}\right)$

$y_p'' = -2Ce^{-2x} - 2Ce^{-2x} + 4Cxe^{-2x}$

$y_p'' = -4Ce^{-2x} + 4Cxe^{-2x}$

$y_p''' = 8Ce^{-2x} + 4Ce^{-2x} - 8Cxe^{-2x}$

$\left[8Ce^{-2x} + 4Ce^{-2x} - 8Cxe^{-2x}\right] + 8\left[Ax + B + \left(Cxe^{-2x}\right)\right] = 2x - 5 + 8e^{-2x}$

$\left[8Ce^{-2x} + 4Ce^{-2x} - 8Cxe^{-2x}\right] + 8Ax + 8B + 8Cxe^{-2x} = 2x - 5 + 8e^{-2x}$

$12Ce^{-2x} + 8Ax + 8B = 2x - 5 + 8e^{-2x} \Rightarrow 12C = 8,\ 8A = 2,\ 8B = -5$

$C = \dfrac{8}{12} \Rightarrow \dfrac{2}{3},\ A = \dfrac{1}{4},\ B = -\dfrac{5}{8}$

$y_p = \dfrac{1}{4}x - \dfrac{5}{8} + \dfrac{2}{3}xe^{-2x}$

$y_p' = \dfrac{1}{4} + \dfrac{2}{3}e^{-2x} - \dfrac{4}{3}xe^{-2x}$

$y_p''' = \dfrac{16}{3}e^{-2x} + \dfrac{8}{3}e^{-2x} - \dfrac{16}{3}xe^{-2x}$

$-5 = c_1 + c_2 - \dfrac{5}{8}$

$3 = -2c_1 + c_2 + \sqrt{3}c_3 + \dfrac{1}{4} + \dfrac{2}{3}$

$-4 = 4c_1 + 2\sqrt{3}c_3 - 2c_2 + 8$