Problem # 1(b) Find the general solution for the following equations by using the method of variation of parameters:

$$x^2y'' - 4xy' + 6y = b(x) = x^4 \sin x, \text{ or, } x^4 \ln x, (x > 0).$$

Solution: This is Euler equation Let $\phi(t) = y(e^t)$, we have

$$[D(D-1) - 4D + 6]\phi(t) = [(D-3)(D-2)]\phi(t) = 0.$$

We derive

$$\phi_1(t) = e^{3t}, \quad \phi_2(t) = e^{2t},$$

or

$$y_1(x) = |x|^3, \quad y_2(x) = |x|^2.$$

As

$$W(y_1, y_2) = \begin{vmatrix} x^3, x^2 \\ 3x^2, 2x \end{vmatrix} = 2x^4 - 3x^4 = -x^4 \neq 0, \quad (x > 0)$$

y_1, y_2 are linearly independent.

To find particular solution, we first write the equation

$$x^2y'' - 4xy' + 6y = x^4 \sin x, \quad (x > 0)$$

in the standard form:

$$y'' - \frac{4}{x}y' + \frac{6}{x^2}y = x^2 \sin x, \quad (x > 0).$$

Letting $y_p = v_1(x)y_1 + v_2(x)y_2$, we may derived that

$$\begin{aligned} y_1 v_1' + y_2 v_2' &= 0, \\ y_1' v_1' + y_2' v_2' &= x^2 \sin x. \end{aligned}$$

With Cramer's rule, we have

$$v_1'(x) = \frac{1}{W(y_1, y_2)} \begin{vmatrix} 0 & x^2 \\ x^2 \sin x & 2x \end{vmatrix} = \frac{-x^4 \sin x}{-x^4} = \sin x;$$

and

$$v_2'(x) = \frac{1}{W(y_1, y_2)} \begin{vmatrix} x^3 & 0 \\ 3x^2 & x^2 \sin x \end{vmatrix} = \frac{x^5 \sin x}{-x^4} = -x \sin x.$$

Hence, we get

$$v_1(x) = \int \sin x \, dx = -\cos x;$$

$$v_2(x) = -\int x \sin x \, dx = x \cos x - \sin x.$$

Finally, we find

$$y_p(x) = y_p = v_1(x)y_1 + v_2(x)y_2 = -x^3 \cos x + x^3 \cos x - x^2 \sin x = -x^2 \sin x.$$

The general solution is:

$$y(x) = C_1 x^3 + C_2 x^2 - x^2 \sin x.$$