Rocket Dynamics and Kinetic Energy Derivation

We start by defining the variables used in our derivation:

- m_i is the initial mass of the rocket, including both the rocket and its fuel (constant).
- R is the rate at which the fuel is burned, in units of mass per time (constant).
- u is the exhaust velocity of the expelled gas relative to the rocket (constant).
- t is the time elapsed.
- $m(t) = m_i - Rt$ is the mass of the rocket at time t.
- $v(t)$ is the velocity of the rocket at time t.
- $f(t)$ is the force (thrust) acting on the rocket at time t.
- $K(t)$ is the kinetic energy of the rocket at time t.

Tsiolkovsky Rocket Equation

The Tsiolkovsky rocket equation relates the change in velocity Δv of a rocket to the exhaust velocity u and the initial and final masses of the rocket. It is given by:

$$\Delta v = u \ln\left(\frac{m_i}{m_f}\right)$$

where: - m_f is the final mass of the rocket after expelling fuel. - m_i is the initial mass of the rocket before expelling fuel.

In our case, $m_f = m_i - Rt$, so the velocity $v(t)$ at time t is:

$$v(t) = u \ln\left(\frac{m_i}{m_i - Rt}\right)$$

Derivation of the Force $f(t)$

The force (thrust) $f(t)$ can be derived from the change in momentum of the rocket. The momentum $p(t)$ of the rocket is given by:

$$p(t) = m(t)v(t)$$

where $m(t) = m_i - Rt$ and $v(t) = u \ln\left(\frac{m_i}{m_i - Rt}\right)$.

Thus:

$$p(t) = (m_i - Rt) \cdot u \ln\left(\frac{m_i}{m_i - Rt}\right)$$

To find the force $f(t)$, we take the time derivative of the momentum $p(t)$:

$$f(t) = \frac{d}{dt} p(t)$$

Applying the product rule:

$$\frac{d}{dt}[m(t)v(t)] = m'(t)v(t) + m(t)v'(t)$$

where:

$$m'(t) = -R$$

$$v'(t) = u \cdot \frac{d}{dt} \ln\left(\frac{m_i}{m_i - Rt}\right) = u \cdot \frac{R}{m_i - Rt}$$

Thus:

$$f(t) = (-R) \cdot u \ln\left(\frac{m_i}{m_i - Rt}\right) + (m_i - Rt) \cdot u \cdot \frac{R}{m_i - Rt}$$

Simplifying:

$$f(t) = -Ru \ln\left(\frac{m_i}{m_i - Rt}\right) + Ru$$

Kinetic Energy Derivation

The kinetic energy $K(t)$ of the rocket at time t is given by:

$$K(t) = \frac{1}{2} m(t) v(t)^2$$

Substituting $m(t) = m_i - Rt$ and $v(t) = u \ln\left(\frac{m_i}{m_i - Rt}\right)$:

$$\boxed{K(t) = \frac{1}{2}(m_i - Rt)\left(u \ln\left(\frac{m_i}{m_i - Rt}\right)\right)^2}$$

Work-Energy Theorem Approach

The total work done by the force $f(t)$ on the rocket and fuel over time t is:

$$W = \int_0^t f(t) \cdot v(t)\, dt$$

Substituting $f(t)$ and $v(t)$:

$$f(t) = Ru - Ru \ln\left(\frac{m_i}{m_i - Rt}\right)$$

$$v(t) = u \ln\left(\frac{m_i}{m_i - Rt}\right)$$

$$W = \int_0^t \left[Ru - Ru \ln\left(\frac{m_i}{m_i - Rt}\right)\right] \cdot u \ln\left(\frac{m_i}{m_i - Rt}\right)\, dt$$

Expanding the integrand:

$$W = Ru \int_0^t u \ln\left(\frac{m_i}{m_i - Rt}\right)\, dt - Ru \int_0^t u \ln^2\left(\frac{m_i}{m_i - Rt}\right)\, dt$$

First Integral

$$W_1 = Ru^2 \int_0^t \ln\left(\frac{m_i}{m_i - Rt}\right)\, dt$$

Substitute $x = m_i - Rt$. Then $dx = -R\, dt$ or $dt = -\frac{dx}{R}$.
When $t = 0$, $x = m_i$. When $t = t$, $x = m_i - Rt$.
The integral becomes:

$$W_1 = -\frac{u^2}{R} \int_{m_i}^{m_i - Rt} \ln\left(\frac{m_i}{x}\right)\, dx$$

Simplify $\ln\left(\frac{m_i}{x}\right)$:

$$\ln\left(\frac{m_i}{x}\right) = \ln(m_i) - \ln(x)$$

Thus:

$$W_1 = -\frac{u^2}{R}\left[\int_{m_i}^{m_i - Rt} \ln(m_i)\, dx - \int_{m_i}^{m_i - Rt} \ln(x)\, dx\right]$$

The first integral is straightforward:

$$\int_{m_i}^{m_i - Rt} \ln(m_i)\, dx = \ln(m_i) \cdot (m_i - Rt - m_i) = -\ln(m_i) \cdot Rt$$

For the second integral:

$$\int_{m_i}^{m_i - Rt} \ln(x)\, dx$$

Use integration by parts where $u = \ln(x)$ and $dv = dx$. Then $du = \frac{1}{x}\, dx$ and $v = x$:

$$\int \ln(x)\, dx = x\ln(x) - \int x \cdot \frac{1}{x}\, dx = x\ln(x) - x$$

Thus:

$$\int_{m_i}^{m_i - Rt} \ln(x)\, dx = [x\ln(x) - x]_{m_i}^{m_i - Rt}$$

$$= (m_i - Rt)\ln(m_i - Rt) - (m_i - Rt) - (m_i \ln(m_i) - m_i)$$

$$= (m_i - Rt)\ln(m_i - Rt) - m_i \ln(m_i) + m_i - (m_i - Rt)$$

$$= (m_i - Rt)\ln(m_i - Rt) - m_i \ln(m_i) + Rt$$

Combining both results:

$$W_1 = -\frac{u^2}{R}\left[-\ln(m_i) \cdot Rt - ((m_i - Rt)\ln(m_i - Rt) - m_i \ln(m_i) + Rt)\right]$$

$$= \frac{u^2}{R}\left[\ln(m_i) \cdot Rt - (m_i - Rt)\ln(m_i - Rt) + m_i \ln(m_i) - Rt\right]$$

Second Integral

$$W_2 = -Ru^2 \int_0^t \ln^2\left(\frac{m_i}{m_i - Rt}\right) dt$$

Substitute $x = m_i - Rt$:

$$W_2 = \frac{u^2}{R} \int_{m_i}^{m_i - Rt} \ln^2\left(\frac{m_i}{x}\right) dx$$

$$\ln^2\left(\frac{m_i}{x}\right) = [\ln(m_i) - \ln(x)]^2$$

$$= \ln^2(m_i) - 2\ln(m_i)\ln(x) + \ln^2(x)$$

Thus:

$$W_2 = \frac{u^2}{R}\left[\int_{m_i}^{m_i - Rt} \ln^2(m_i)\, dx - 2\ln(m_i) \int_{m_i}^{m_i - Rt} \ln(x)\, dx + \int_{m_i}^{m_i - Rt} \ln^2(x)\, dx\right]$$

The first integral:

$$\int_{m_i}^{m_i-Rt} \ln^2(m_i)\,dx = \ln^2(m_i)\cdot(m_i - Rt - m_i) = -\ln^2(m_i)\cdot Rt$$

For the second term (already computed):

$$\int_{m_i}^{m_i-Rt} \ln(x)\,dx = (m_i - Rt)\ln(m_i - Rt) - m_i\ln(m_i) + Rt$$

For the third integral, use integration by parts:

$$\int \ln^2(x)\,dx = x\ln^2(x) - 2\int x\ln(x)\cdot\frac{1}{x}\,dx = x\ln^2(x) - 2\int \ln(x)\,dx$$

$$= x\ln^2(x) - 2(x\ln(x) - x) = x\ln^2(x) - 2x\ln(x) + 2x$$

Thus:

$$\int_{m_i}^{m_i-Rt} \ln^2(x)\,dx = \left[x\ln^2(x) - 2x\ln(x) + 2x\right]_{m_i}^{m_i-Rt}$$

$$= (m_i - Rt)\ln^2(m_i - Rt) - 2(m_i - Rt)\ln(m_i - Rt) + 2(m_i - Rt) - \left[m_i\ln^2(m_i) - 2m_i\ln(m_i) + 2m_i\right]$$

$$= (m_i - Rt)\ln^2(m_i - Rt) - m_i\ln^2(m_i) - 2(m_i - Rt)\ln(m_i - Rt) + 2m_i\ln(m_i) + 2Rt$$

Combining all results:

$$W_2 = \frac{u^2}{R}\left[-\ln^2(m_i)\cdot Rt - 2\ln(m_i)\left[(m_i - Rt)\ln(m_i - Rt) - m_i\ln(m_i) + Rt\right] + \text{expression for } \ln^2(x) \text{ integral}\right]$$

Combining W_1 and W_2

$$W = W_1 + W_2$$

Substituting the results of W_1 and W_2:

$$W = \frac{u^2}{R}\left[\ln(m_i)\cdot Rt - (m_i - Rt)\ln(m_i - Rt) + m_i\ln(m_i) - Rt\right]$$

$$+\frac{u^2}{R}\left[(m_i - Rt)\ln^2(m_i - Rt) - m_i\ln^2(m_i) - 2(m_i - Rt)\ln(m_i - Rt) + 2m_i\ln(m_i) + 2Rt\right]$$

Combining like terms and simplifying:

$$W = \frac{u^2}{R}\left[(m_i - Rt)\ln^2(m_i - Rt) - m_i \ln^2(m_i) + 2m_i \ln(m_i) - (m_i - Rt)\ln(m_i - Rt) + 2Rt\right]$$

$$+\frac{u^2}{R}\left[-(m_i - Rt)\ln(m_i - Rt) + m_i \ln(m_i) - Rt + 2Rt\right]$$

Thus:

$$W = \frac{u^2}{R}\left[(m_i - Rt)\ln^2(m_i - Rt) - m_i \ln^2(m_i) + 2m_i \ln(m_i) - (m_i - Rt)\ln(m_i - Rt) + 2Rt\right]$$

The total work done W is given by:

$$\boxed{W = \frac{u^2}{R}\left[(m_i - Rt)\ln^2(m_i - Rt) - m_i \ln^2(m_i) + 2m_i \ln(m_i) - (m_i - Rt)\ln(m_i - Rt) + 2Rt\right]}$$

After this, I would expect that the addition of the Kinetic Energy of the fuel

$$d(KE_{Fuel}) = \frac{1}{2}m(u - v(t))^2$$

and the KE of the rocket would equal the total work, but they do not.